Fusion of Positive Energy Representations of $\mathbf{LSpin_{2n}}$

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To Corinne,
and my parents
for their biblical patience

Preface

The contents of this dissertation are original, except where explicit reference is made to the work of others. The original material is the result of my own work and includes nothing which is the outcome of a collaborative effort. No part of this dissertation has been, or is currently being submitted for a degree, diploma or qualification at this or any other University.

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Contents

Introduction

Let G be a compact, connected and simply–connected simple Lie group. The *loop group* $LG = C^\infty(S^1, G)$ admits a class of projective unitary representations, those of *positive energy*, whose theory parallels that of the representations of G. In particular, they are completely reducible and the irreducible ones are described by the same data as those of G together with an additional parameter, the *level* $\ell \in \mathbb{N}$ which classifies the underlying projective cocycle. Morevoer, only finitely many irreducibles exist at a given level ℓ.

In their pioneering work, Belavin–Polyakov–Zamolodchikov [**BPZ**] and Knizhnik–Zamolodchikov [**KZ**] associate to each positive energy representation a quantum *primary field*. They introduce the notion of *fusion* of primary fields, based on their short range *operator product expansion*. Their computations implicitly suggest the existence of a tensor product operation, differing from the usual one, between positive energy representations at a fixed level which should give them a structure similar to the representation ring of a finite group. The fundamental problem of giving a mathematically sound definition of this product which is associative and reflects the computations of the physicists has proved to be difficult so far. Several independent, and at present unrelated definitions have been suggested by Segal, Kazhdan–Lusztig, Jones–Wassermann, Borcherds, Beilinson–Drinfeld and Huang–Lepowsky.

The aim of the present dissertation is to give such a definition for the loop group of $G = \mathrm{Spin}_{2n}$, $n \geq 3$ and to characterise the resulting algebraic structure on the category $\mathcal{P}_\ell$ of positive energy representations at level ℓ. Conjecturally, the latter is described by the *Verlinde rules* [**Ve**], that is as a quotient of the representation ring of Spin_{2n} and we establish a number of results in agreement with this. In particular, we compute explicitly the fusion of the vector representation of $L\,\mathrm{Spin}_{2n}$, *i.e.* the one corresponding to the defining representation of SO_{2n} with the positive energy representations corresponding to single–valued representations of SO_{2n} and, at odd level with all representations. Moreover, we prove that the level 1 fusion ring of $L\,\mathrm{Spin}_{2n}$ is isomorphic to the group algebra of the centre of Spin_{2n}. As A. Wassermann has recently indicated, our techniques should extend to yield the complete structure of the fusion ring of $L\,\mathrm{Spin}_{2n}$. Together with a study of the category theoretic properties of $\mathcal{P}_\ell$, this would lead to the first rigorous definition of the three–manifold invariants defined by link surgery corresponding to the group Spin_{2n}.

Our study is modelled on the von Neumann algebra approach to fusion, first introduced by Wassermann in relation to the fusion ring of $L\,\mathrm{SU}_n$ [**Wa1, Wa2, Wa3, Wa4, J4**]. The latter originates in a joint program with V. Jones aimed at understanding unitary conformal field theories from the point of view of operator algebras which arose from Jones' suggestion that there might be a subfactor explanation for the coincidence of certain braid group representations that appeared in statistical mechanics and conformal field theory [**J2, J3**]. It relies on the use of *Connes fusion*, a tensor product operation on bimodules over von Neumann algebras originally developed by Connes [**Co, Sa**]. The link with loop groups arises by regarding positive energy representations as bimodules over the groups $L_I\,\mathrm{Spin}_{2n}, L_{I^c}\,\mathrm{Spin}_{2n}$ of loops supported in a proper interval $I \subset S^1$ and its complement and leads to a definition of fusion which is manifestly unitary and associative. The use of Connes fusion requires one to check that positive energy representations at a given level satisfy the axioms of locality, Haag duality and local equivalence. This amounts in essence to showing that $\mathcal{P}_\ell$ constitutes a quantum field theory in the sense of Doplicher–Haag–Roberts and provides interesting examples of such theories in 1+1 dimensions since, as pointed out by Goddard, Nahm and Olive [**GNO**] they are distinct from

free field theories for all but finitely many values of ℓ.

Wassermann showed that the intertwiners necessary to compute Connes fusion could be obtained by smearing the primary fields of Knizhnik and Zamolodchikov, provided their continuity as operator–valued distributions was established. With this analysis done, the actual computation of fusion depends on the commutation or *braiding* properties of the primary fields through *braiding–fusion duality*. More precisely, as established in [**KZ, TK1**], the structure constants governing the braiding of primary fields arise as the entries of the analytic continuation matrix of a vector–valued fuchsian ODE, the *Knizhnik–Zamolodchikov equation*, from the singular point 0 to ∞. The computability of fusion therefore rests on the solvability of this ODE. Wassermann's method was successfully applied by Loke [**Lo**] to the positive energy representations of the diffeomorphism group of the circle.

Another important input in this study is the theory of superselection sectors developed within the context of algebraic quantum field theory by Doplicher, Haag and Roberts [**DHR1, DHR2**]. After the laborious analysis involved in explicitly checking a few fusion rules, it follows fairly easily that the properties required to apply the DHR theory hold. One is therefore guaranteed the existence of a quantum or statistical dimension, of a Markov trace and an action of the braid group whose use plays an important rôle in deducing the remaining fusion rules.

As mentioned above, an important precedent for the present study is Wassermann's computation of the fusion ring of $L\,\mathrm{SU}_n$. This constituted an invaluable conceptual framework and the backbone on which this dissertation rests. We wish however to stress that, much as the invariant theory of Spin_{2n} differs from that of SU_n, the study of the loop groups of the spin groups poses a whole new layer of conceptual and technical difficulties. We sketch them here, in technical jargon and refer to the outline of our results given in the next section or to the main text for more details.

With a degree of oversimplification, one may say that most analytic results needed for the positive energy representations of $L\,\mathrm{SU}_n$ can be derived from their fermionic construction. This realises the level 1 representations as summands of the basic representation of LU_n obtained from that of the Clifford algebra of $\mathcal{H} = L^2(S^1, \mathbb{C}^n)$ on the Fock space $\mathcal{F} = \Lambda\mathcal{H}_+$ of the Hardy space $\mathcal{H}_+$. The level ℓ representations are then realised inside the ℓ–fold tensor product $\mathcal{F}^{\otimes\ell}$, itself a Fock space representation of the Clifford algebra of $L^2(S^1, \mathbb{C}^{n\ell})$. This uniform construction simplifies the proof of the following points

(i) All positive energy representations of $L\,\mathrm{SU}_n$ at level ℓ give rise to the same projective cocycle. This is obvious since they are summands of a single representation.
(ii) Positive energy representations at equal level are unitarily equivalent for the local loop groups $L_I\,\mathrm{SU}_n$. This follows from the (non–trivial) fact that these generate type III factors in $\mathcal{F}^{\otimes\ell}$ and that all projections in such factors are equivalent.
(iii) All primary fields extend to unbounded operator–valued distributions. This simply follows because they may be realised as products of Fermi fields compressed by projections, although checking that this procedure yields all primary fields is a subtler matter.

By contrast,

(i) The level 1 representations of $L\,\mathrm{Spin}_{2n}$ appear, grouped by two as summands of two distinct Fock spaces, the Neveu–Schwarz and the Ramond sector. This separation obscures (i) above, the proof of which requires the use of the spin primary field which links the two sectors, or of analytic vector techniques.
(ii) The local equivalence at level 1 must be established by the use of the outer automorphic action of the centre of Spin_{2n} on $L\,\mathrm{Spin}_{2n}$ via conjugation by discontinuous loops.
(iii) The fermionic construction for $L\,\mathrm{Spin}_{2n}$ only gives information on the vector primary field but little or none on the spin fields. We remedy this problem by making use of the vertex operator model of Segal and Kac–Frenkel to give an explicit construction of all the level 1

primary fields in the bosonic picture. A technique due to Wassermann then allows to establish their continuity properties as well as those of a number of higher level primary fields.

Another substantial difference lies in the study of the relevant Knizhnik–Zamolodchikov equations. As well–known, solutions of the KZ equations for all simple Lie algebras have been given by Schechtman and Varchenko [**SV**] but their combinatorial complexity makes them intractable for computational purposes. Somewhat miraculously, the computation of the fusion of a general representation of $L\,\mathrm{SU}_n$ with the vector representation may be obtained from a detailed study of the monodromy properties of the generalised hypergeometric function studied by Thomae at the end of the 19th century [**Wa3**]. No such classical treatment was available to us, but a similar miracle occurs for $L\,\mathrm{Spin}_{2n}$. The 3rd order ODE required to compute the fusion of the vector representation with its symmetric powers reduces to the Dotsenko–Fateev equation [**DF**] discovered by the physicists in the context of minimal models. The analysis of the first part of the dissertation then enables us to establish a small number of explicit fusion rules together with upper bounds for the multiplicities of the general fusion rules with the vector representation. The computation is concluded by resorting to the algebraic techniques of quantum invariant theory, and in particular to the action of the braid group coming from the Doplicher–Haag–Roberts theory. The corresponding representations can be handled using some algebraic techniques of Wenzl connected with the Birman–Wenzl algebra [**We2, We3**]. The fusion rules are finally pinned down by interpreting the quantum dimension of DHR in the context of Perron–Frobenius theory.

1. Outline of contents

We outline below the contents of this dissertation, developing only what is minimally necessary to sketch a proof of our main results. A detailed account of each chapter may be found in its opening paragraphs. §1.1 describes the classification of positive energy representations of the loop group of Spin_{2n}. In §1.2, we define the Connes fusion of two such representations and indicate a scheme to compute it based upon the commutation properties of certain intertwiners. The latter are derived in §1.3 from the braiding properties of primary fields. These are the building blocks of conformal field theory and we outline their definition and main properties. Our main results are given in §1.4.

1.1. Positive energy representations of LG.

Let $G=\mathrm{Spin}_{2n}$, $n\geq 3$ be the universal covering group of SO_{2n} and $\mathfrak{g}=\mathfrak{so}_{2n}$ its Lie algebra. The irreducible representations of G are classified by their highest weight, a sequence $\zeta_1\geq\cdots\geq\zeta_{n-1}\geq|\zeta_n|$ with $\zeta_i\in\mathbb{Z}$ for any i or $\zeta_i\in\frac{1}{2}+\mathbb{Z}$ for any i corresponding respectively to the single and two–valued representations of SO_{2n}.

Consider now the *loop group* $LG=C^\infty(S^1,G)$ of G with Lie algebra $L\mathfrak{g}=C^\infty(S^1,\mathfrak{g})$. Both are acted upon by $\mathrm{Rot}\,S^1$, the group of rotations of the circle, via reparametrisation. The representations of LG have a similar classification, if one restricts one's attention to those of *positive energy*. These are projective unitary representations $\pi:LG\rightarrow PU(\mathcal{H})$ extending to the semi–direct product $LG\rtimes\mathrm{Rot}\,S^1$ in such a way that the infinitesimal generator of rotations is bounded below and has finite–dimensional eigenspaces. In other words, $\mathcal{H}=\bigoplus_{n\geq 0}\mathcal{H}(n)$ where $\mathcal{H}(n)=\{\xi\in\mathcal{H}|\pi(R_\theta)\xi=e^{in\theta}\xi\}$, the *subspace of energy* n, supports a finite–dimensional representation of the subgroup of constant loops $G\subset LG$.

Positive energy representations are completely reducible and their classification is obtained in the following way. As explained in chapter I, the *finite energy subspace* $\mathcal{H}^{\mathrm{fin}}$ of a positive energy representation $\mathcal{H}$, that is the algebraic direct sum $\bigoplus_{n\geq 0}\mathcal{H}(n)$ supports a projective representation of $L^{\mathrm{pol}}\mathfrak{g}$, the dense subalgebra of $L\mathfrak{g}$ consisting of $\mathfrak{g}$–valued trigonometric polynomials, and therefore one of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ corresponding to $\mathfrak{g}$. The latter is the semi–direct product $\widetilde{\mathfrak{g}}\rtimes\mathbb{C}d$

of the universal central extension $\widetilde{\mathfrak{g}}$ of $L^{\mathrm{pol}}\mathfrak{g}$ by the infinitesimal action of rotations. Moreover, the classification of $\mathcal{H}$ as an LG–module is equivalent to that of the $\widetilde{\mathfrak{g}}$–module $\mathcal{H}^{\mathrm{fin}}$. In particular, an irreducible $\mathcal{H}$ is uniquely determined by an integer ℓ called the *level*, which classifies the Lie algebra cocyle on $L^{\mathrm{pol}}\mathfrak{g}$ and its *lowest energy subspace* $\mathcal{H}(0)$, an irreducible G–module whose highest weight ζ is bound by the requirement that $\zeta_1 + \zeta_2 \leq \ell$. An irreducible G–module satisfying this constraint is called *admissible at level* ℓ and always arises as the lowest energy subspace of a (necessarily unique) irreducible positive energy representation at level ℓ.

The centre $Z(G)$ of G acts by outer automorphisms on LG via conjugation by *discontinuous loops*, *i.e.* lifts to G of loops in $G/Z(G)$. As shown in chapter I, this induces a level preserving action of $Z(G)$ on the positive energy representations of LG. In fact, if $z \in Z(G)$ and $\mathcal{H}$ is an irreducible positive energy representation, the lowest energy subspace of the conjugated representation $z\mathcal{H}$ may be characterised explicitly in terms of the level ℓ, the lowest energy subspace $\mathcal{H}(0)$ and z by realising $Z(G)$ as a distinguished subgroup of the automorphisms of the extended Dynkin diagram of G. This shows in particular that at level 1, the action of $Z(G)$ is transitive and free.

Chapter II is devoted to the analytic properties of positive energy representations and in particular to the construction of a dense subspace on which both LG and its Lie algebra $L\mathfrak{g}$ act, which is required as a natural domain for the smeared primary fields. These are densely defined LG–intertwiners mapping between different positive energy representations and do not in general extend to bounded maps. If $\mathcal{H}$ is a positive energy representation, such a domain is provided by the subspace of *smooth vectors* $\mathcal{H}^\infty$ for $\mathrm{Rot}\, S^1$, *i.e.* those vectors whose projection onto the subspace of energy n decreases in norm faster than any polynomial in n.

1.2. Connes fusion of positive energy representations.

The notion of *Connes fusion* of positive energy representations of LG arises by regarding them as bimodules over the subgroups $L_IG, L_{I^c}G$ of loops supported in a given interval $I \subset S^1$ and its complement and using a tensor product operation on bimodules over von Neumann algebras developed by Connes [**Co, Sa**].

Recall that a bimodule $\mathcal{H}$ over a pair (M, N) of von Neumann algebras is a Hilbert space supporting commuting representations of M and N. To any two bimodules X, Y over the pairs (M, N), $(\widetilde{N}, P)$, Connes fusion associates an (M, P)–bimodule denoted by $X \boxtimes Y$. The definition of $X \boxtimes Y$ relies on, but is ultimately independent of the choice of a reference or *vacuum* $(N, \widetilde{N})$–bimodule $\mathcal{V}$ with a cyclic vector Ω for both actions and for which *Haag duality* holds, *i.e.* the actions of N and $\widetilde{N}$ are each other's commutant. Given $\mathcal{V}$, we form the intertwiner spaces

$$\mathfrak{X} = \mathrm{Hom}_N(\mathcal{V}, X) \qquad \text{and} \qquad \mathfrak{Y} = \mathrm{Hom}_{\widetilde{N}}(\mathcal{V}, Y) \tag{1.2.1}$$

and consider the sequilinear form on the algebraic tensor product $\mathfrak{X} \otimes \mathfrak{Y}$ given by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = (x_2^* x_1 y_2^* y_1 \Omega, \Omega) \tag{1.2.2}$$

where the inner product on the right hand–side is taken in $\mathcal{V}$. If $x_1 = x_2$ and $y_1 = y_2$, Haag duality implies that $x_2^* x_1$ and $y_2^* y_1$ are commuting positive operators and therefore that $\langle \cdot, \cdot \rangle$ is positive semi–definite. By definition, the bimodule $X \boxtimes Y$ is the Hilbert space completion of $\mathfrak{X} \otimes \mathfrak{Y}$ with respect to $\langle \cdot, \cdot \rangle$, with (M, P) acting as $(m, p)x \otimes y = mx \otimes py$.

Applying the above to the positive energy representations of LG requires a number of preliminary results which are established in chapters III and IV. Let $\mathcal{P}_\ell$ be the set of positive energy representations at a fixed level ℓ. We wish to regard any $(\mathcal{H}, \pi) \in \mathcal{P}_\ell$ as a bimodule over the pair $\pi_0(L_IG)'', \pi_0(L_{I^c}G)''$ where π_0 is the *vacuum representation* at level ℓ whose lowest energy subspace is, by definition the trivial G–module. The well–foundedness of this change of perspective is justified by the following properties

(i) *Locality* : $\pi(L_IG)'' \subset \pi(L_{I^c}G)'$ for any $(\pi,\mathcal{H}) \in \mathcal{P}_\ell$. In other words, $\mathcal{H}$ is a $(\pi(L_IG)^{''},\pi(L_{I^c}G)^{''})$–bimodule.

(ii) *Local equivalence* : All $(\pi,\mathcal{H}) \in \mathcal{P}_\ell$ are unitarily equivalent as L_IG–modules. Thus we may unambiguously identify $\pi(L_IG)''$ with $\pi_0(L_IG)''$ and consider $\mathcal{H}$ as a $(\pi_0(L_IG)'',\pi_0(L_{I^c}G)'')$–bimodule.

(iii) *von Neumann density* : $\pi(L_IG) \times \pi(L_{I^c}G)$ is strongly dense in $\pi(LG)$. Thus, inequivalent irreducible positive energy representations of LG remain so when regarded as bimodules.

The rôle of the reference bimodule is played by the vacuum representation $(\pi_0,\mathcal{H}_0)$. Two crucial facts need to be established in this respect

(iv) *Reeh-Schlieder theorem* : any finite energy vector of a positive energy representation π is cyclic under $\pi(L_IG)$. In particular, the lowest energy vector $\Omega \in \mathcal{H}_0(0)$ is cyclic for $\pi_0(L_IG)''$ and $\pi_0(L_{I^c}G)''$.

(v) *Haag duality* : $\pi_0(L_IG)^{''} = \pi_0(L_{I^c}G)^{'}$.

Finally, another technically crucial property of the algebras $\pi(L_IG)''$ is the following

(vi) *Factoriality* : The algebras $\pi(L_IG)''$ with $I \subsetneq S^1$ are type III_1 factors.

Once properties (i)-(vi) are established, the definition of Connes fusion may be adapted to our setting. Let $\mathcal{H}_i,\mathcal{H}_j \in \mathcal{P}_\ell$ and form the intertwiner spaces

$$\mathfrak{X}_i = \mathrm{Hom}_{L_{I^c}G}(\mathcal{H}_0,\mathcal{H}_i) \qquad\qquad \mathfrak{Y}_j = \mathrm{Hom}_{L_IG}(\mathcal{H}_0,\mathcal{H}_j) \tag{1.2.3}$$

Then, $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is the completion of $\mathfrak{X}_i \otimes \mathfrak{Y}_j$ with respect to the inner product (1.2.2). $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is manifestly unitary and the functorial properties of Connes fusion imply that $\boxtimes$ is associative. It is not *a priori* clear however that $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is of positive energy. In fact there is no naturally defined action of LG on it, let alone an intertwining one of $\mathrm{Rot}\, S^1$. These facts will be checked a posteriori by explicitly computing the fusion.

Consider now a first attempt at computing $\mathcal{H}_i \boxtimes \mathcal{H}_j$. Notice that, if $y_{j0} \in \mathrm{Hom}_{L_IG}(\mathcal{H}_0,\mathcal{H}_j)$, then

$$y_{j0}^* y_{j0} \in \mathrm{Hom}_{L_IG}(\mathcal{H}_0,\mathcal{H}_0) = \pi_0(L_IG)' \tag{1.2.4}$$

By Haag duality, $y_{j0}^* y_{j0}$ lies in $\pi_0(L_{I^c}G)''$ and may therefore be represented, via local equivalence, on $\mathcal{H}_i$. Thus, if $x_{i0} \in \mathfrak{X}_i$ so that $x_{i0}\pi_0(\gamma) = \pi_i(\gamma)x_{i0}$ for any $\gamma \in L_{I^c}G$, then

$$x_{i0}y_{j0}^* y_{j0} = \pi_i(y_{j0}^* y_{j0})x_{i0} \tag{1.2.5}$$

Much of our efforts will be aimed at establishing a *transport formula* giving an explicit expression for $\pi_i(y_{j0}^* y_{j0})$, that is the identity

$$\pi_i(y_{j0}^* y_{j0}) = \sum_k \beta_k y_{ki}^* y_{ki} \tag{1.2.6}$$

where the β_k are some positive constants labelled by inequivalent positive energy representations $\pi_k \in \mathcal{P}_\ell$ and the $y_{ki} \in \mathrm{Hom}_{L_IG}(\mathcal{H}_i,\mathcal{H}_k)$ depend linearly on y_{j0}. Deferring to §1.3 an explanation of why (1.2.6) should hold at all, notice that it allows to compute $\mathcal{H}_i \boxtimes \mathcal{H}_j$. Indeed, by (1.2.5) and (1.2.6)

$$\begin{aligned}\|x_{i0} \otimes y_{j0}\|^2 &= (x_{i0}^* x_{i0} y_{j0}^* y_{j0}\Omega,\Omega)\\ &= (x_{i0}^*\pi_i(y_{j0}^* y_{j0})x_{i0}\Omega,\Omega)\\ &= \sum_k \beta_k (x_{i0}^* y_{ki}^* y_{ki} x_{i0}\Omega,\Omega)\\ &= \|\bigoplus_k \beta_k^{\frac{1}{2}} y_{ki}x_{i0}\Omega\|^2\end{aligned} \tag{1.2.7}$$

and therefore the map

$$U : \mathfrak{X}_i \otimes \mathfrak{Y}_j \rightarrow \bigoplus_k \mathcal{H}_k,\ x_{i0} \otimes y_{j0} \rightarrow \bigoplus_k \beta_k^{\frac{1}{2}} y_{ki}x_{i0}\Omega \tag{1.2.8}$$

extends to an isometry $\mathcal{H}_i \boxtimes \mathcal{H}_j \to \bigoplus_k \mathcal{H}_k$ which is easily seen to be $L_I G \times L_{I^c} G$–equivariant. Since the $\mathcal{H}_k$ are irreducible and inequivalent $L_I G \times L_{I^c} G$–modules, and the image of U has non–zero intersection with each of the $\mathcal{H}_k$, it is dense by Shur's lemma and we conclude that

$$\mathcal{H}_i \boxtimes \mathcal{H}_j = \bigoplus_k \mathcal{H}_k \tag{1.2.9}$$

In particular, $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is of positive energy or, more precisely, arises as the restriction to $L_I G \times L_{I^c} G$ of a positive energy LG–module [1].

To conclude, notice that the transport formula only needs to be established for one unitary $y_{j0} \in \mathfrak{Y}_j$. Such an element certainly exists by local equivalence. Moreover, by Haag duality,

$$\mathfrak{Y}_j = \mathrm{Hom}_{L_I G}(\mathcal{H}_0, \mathcal{H}_j) = y_{j0}\pi_0(L_I G)' = y_{j0}\pi_0(L_{I^c} G)'' \tag{1.2.10}$$

Thus, if (1.2.6) holds for y_{j0} and a lies in the *–algebra generated by $\pi_0(L_{I^c} G)$, we have for $y = y_{j0} a$

$$\pi_i(y^* y) = \pi_i(a^* y_{j0}^* y_{j0} a) = \pi_i(a)^* \pi_i(y_{j0}^* y_{j0}) \pi_i(a) = \sum_k \beta_k \pi_i(a)^* y_{ki}^* y_{ki} \pi_i(a) \tag{1.2.11}$$

and the transport formula holds for y.

1.3. Primary fields and their braiding properties.

Let $\mathcal{H}_i, \mathcal{H}_j$ be irreducible positive energy representations at level ℓ with lowest energy subspaces V_i, V_j. Let V_k be an irreducible G–module admissible at level ℓ and $V_k[z, z^{-1}]$ the space of V_k–valued trigonometric polynomials. $L^{\mathrm{pol}}\mathfrak{g}$ acts on $V_k[z, z^{-1}]$ by multiplication and $\mathrm{Rot}\, S^1$ by $R_\theta v \otimes z^n = e^{-in\theta} v \otimes z^n$. A *primary field* of charge V_k is a linear map

$$\phi_{ji}^k : \mathcal{H}_i^{\mathrm{fin}} \otimes V_k[z, z^{-1}] \to \mathcal{H}_j^{\mathrm{fin}} \tag{1.3.1}$$

intertwining the actions of $L^{\mathrm{pol}}\mathfrak{g} \rtimes \mathrm{Rot}\, S^1$. ϕ_{ji}^k may be regarded as an endomorphism–valued algebraic distribution associating to any $f \in V_k[z, z^{-1}]$ the *smeared field* $\phi_{ji}^k(f) \in \mathrm{Hom}(\mathcal{H}_i^{\mathrm{fin}}, \mathcal{H}_j^{\mathrm{fin}})$ and is best represented by its generating function

$$\phi_{ji}^k(z) = \phi_{ji}^k(v, z) = \sum_{n \in \mathbb{N}} \phi_{ji}^k(v, n) z^{-n} \tag{1.3.2}$$

where $\phi_{ji}^k(v, n) = \phi_{ji}^k(v \otimes z^n)$, $v \in V_k$ [2]. By restriction, ϕ_{ji}^k defines a finite–dimensional G–intertwiner or *initial term* $\varphi : V_i \otimes V_k \to V_j$ which determines ϕ_{ji}^k uniquely. Thus, the space of ϕ_{ji}^k is a subspace of $\mathrm{Hom}_G(V_i \otimes V_k, V_j)$ and is therefore finite–dimensional.

The relevance of primary fields to the study of fusion lies in the fact that they may be used to construct intertwiners for the local loop groups and ultimately explicit elements in (1.2.3) by smearing them against functions supported in a disjoint interval. This involves extending them to operator–valued distributions on $C^\infty(S^1, V_k)$ and requires a careful study of their continuity properties which is carried out in chapters V and VI. This shows that they extend to jointly continuous maps $\mathcal{H}_i^\infty \otimes C^\infty(S^1, V_k) \to \mathcal{H}_j^\infty$, where $\mathcal{H}_i^\infty \subset \mathcal{H}_i, \mathcal{H}_j^\infty \subset \mathcal{H}_j$ are the subspaces of smooth vectors satisfying as expected

$$\pi_j(\gamma) \phi_{ji}^k(f) \pi_i(\gamma)^* = \phi_{ji}^k(\gamma f) \tag{1.3.3}$$

for any $\gamma \in LG$. Choosing f supported in I^c yields an operator $\mathcal{H}_i^\infty \to \mathcal{H}_j^\infty$ commuting with $L_I G$. Although in general $\phi(f)$ is unbounded, a bounded operator may be obtained by taking the phase of its polar decomposition. This yields an element of $\mathrm{Hom}_{L_I G}(\mathcal{H}_i, \mathcal{H}_j)$.

[1] In the foregoing, we have implicitly restricted our attention to the case where the transport formula does not involve any multiplicities and therefore the summands of $\mathcal{H}_i \boxtimes \mathcal{H}_j$ have multiplicity one. The general case follows in a similar way.

[2] For the *cognoscenti*, we shall abusively denote by $\phi_{ji}^k(z)$ both the integrally moded primary field and $\sum_{n \in \mathbb{Z}} \phi_{ji}^k(v, n) z^{-n-\Delta}$ where Δ is the conformal dimension of ϕ_{ji}^k.

The transport formula (1.2.6) for these elements is derived from the remarkable commutation or *braiding* properties of primary fields to which we now turn. These read

$$\phi^{i_3}_{i_4 j}(w)\phi^{i_2}_{j i_1}(z) = \sum_h \lambda_h \phi^{i_2}_{i_4 h}(z)\phi^{i_3}_{h i_1}(w) \tag{1.3.4}$$

and are a generalisation of the canonical commutation or anti–commutation relations $\phi(z)\phi(w) = \pm\phi(w)\phi(z)$ to a setting where the symmetric group $\mathfrak{S}_2$ is replaced by the braid group on two strings, *i.e.* $\mathbb{Z}$. The sum on the right hand–side spans all level ℓ irreducible positive energy representations $\mathcal{H}_h$ and all primary fields

$$\phi^{i_3}_{h i_1} : \mathcal{H}^{\text{fin}}_{i_1} \otimes V_{i_3}[z, z^{-1}] \to \mathcal{H}^{\text{fin}}_h \qquad \text{and} \qquad \phi^{i_2}_{i_4 h} : \mathcal{H}^{\text{fin}}_h \otimes V_{i_2}[z, z^{-1}] \to \mathcal{H}^{\text{fin}}_{i_4} \tag{1.3.5}$$

Note that (1.3.4) is not a formal power series identity. Rather, it should be interpreted as saying that the matrix coefficients of the right hand–side define a holomorphic function on $|z| > |w|$. Similarly, those of the left hand–side define a holomorphic function on $|w| > |z|$ which may be analytically continued to a (multi–valued) function on $z \neq w$ and coincides with the right–hand side on $|w| > |z|$.

Braiding identities such as (1.3.4) are established by means of the *four–point function* of the primary fields

$$F = (\phi^{i_3}_{i_4 j}(v_3, w)\phi^{i_2}_{j i_1}(v_2, z)v_1, v_4) \tag{1.3.6}$$

where $v_1 \in V_{i_1} = \mathcal{H}_{i_1}(0)$ and $v_4 \in V_{i_4} = \mathcal{H}_{i_4}(0)$. F is a formal power series in z, w with coefficients in $V_{i_4} \otimes V^*_{i_3} \otimes V^*_{i_2} \otimes V^*_{i_1}$, and uniquely determines the product $\phi^{i_3}_{i_4 j}(w)\phi^{i_2}_{j i_1}(z)$. An important feature of F is that it satisfies the *Knizhnik–Zamolodchikov equations* with respect to the variable $\zeta = zw^{-1}$

$$\frac{dF}{d\zeta} = \frac{1}{\kappa}\Big(\frac{\Omega_{12}}{\zeta} + \frac{\Omega_{23}}{\zeta - 1}\Big)F \tag{1.3.7}$$

where $\kappa = \ell + 2(n-1)$ and the Ω_{ij} are matrices canonically associated to the tensor product $V_{i_4} \otimes V^*_{i_3} \otimes V^*_{i_2} \otimes V^*_{i_1}$. The above is a fuchsian ODE with regular singular points at $0, 1, \infty$ and, as explained in chapter VII, the four–point functions of products $\phi^{i_3}_{i_4 j}(w)\phi^{i_2}_{j i_1}(z)$ with variable j or products $\phi^{i_2}_{i_4 h}(w)\phi^{i_3}_{h i_1}(z)$ with variable h form a basis of solutions of these equations [3] so that (1.3.4) holds. In fact, the solutions corresponding to the products $\phi^{i_3}_{i_4 j}(w)\phi^{i_2}_{j i_1}(z)$ diagonalise the monodromy of (1.3.7) about the singular point 0 while those corresponding to $\phi^{i_2}_{i_4 h}(w)\phi^{i_3}_{h i_1}(z)$ diagonalise the monodromy about ∞. It follows that the braiding constants λ_h may be computed by analytically continuing the four–point function (1.3.6) from 0 to ∞ and re–expressing it as a sum of functions diagonalising the monodromy at ∞.

The smeared primary fields obey the same braiding relations as their algebraic counterparts. Fix for definiteness $I = (0, \pi)$ so that $I^c = (\pi, 2\pi)$ and let $f \in C^\infty(S^1, V_{i_3})$ and $g \in C^\infty(S^1, V_{i_2})$ be supported in I and I^c respectively. Then, as shown in chapter VII, the following holds on $\mathcal{H}^\infty_{i_1}$

$$\phi^{i_3}_{i_4 j}(f)\phi^{i_2}_{j i_1}(g) = \sum_h \beta_h \phi^{i_2}_{i_4 h}(g e_{\alpha_h})\phi^{i_3}_{h i_1}(f e_{-\alpha_h}) \tag{1.3.8}$$

where $e_\mu(\theta) = e^{i\mu\theta}$ and the α_{jh} are inessential phase corrections.

We now outline the proof of the transport formula (1.2.6). We start from the braiding relation

$$\phi^i_{i0}(w)\phi^{\overline{j}}_{0j}(z) = \sum_k \lambda_k \phi^{\overline{j}}_{ik}(z)\phi^i_{kj}(w) \tag{1.3.9}$$

Here, $\phi^i_{i0}(w)$ and $\phi^{\overline{j}}_{0j}(z)$ are the primary fields whose initial terms are the canonical intertwiners $V_i \otimes \mathbb{C} \to V_i$ and $V_j \otimes V^*_j \to \mathbb{C}$. k labels the irreducible summands of $V_j \otimes V_i$ and we assume for

[3] In fact, of a canonical subspace of these.

simplicity that these have multiplicity one so that for any such k there are at most a primary field of the form ϕ^i_{kj} and one of the form $\phi^{\overline{j}}_{ik}$. We shall need another braiding relation, namely

$$\phi^i_{kj}(w)\phi^j_{j0}(z) = \epsilon_k \phi^j_{ki}(z)\phi^i_{i0}(w) \tag{1.3.10}$$

where the sum on the right hand–side of (1.3.10) contains only one term since there are only one primary field of the form ϕ^i_{i0} and one of the form ϕ^j_{ki}. Let now $f \in C^\infty(S^1, V_i)$ and $g \in C^\infty(S^1, V_j)$ so that $\overline{g} \in C^\infty(S^1, V_j^*)$ be supported in I and I^c respectively. Then, smearing (1.3.9)–(1.3.10), we find

$$\phi^i_{i0}(f)\phi^{\overline{j}}_{0j}(\overline{g}) = \sum_k \lambda_k \phi^{\overline{j}}_{ik}(\overline{g}e_{\alpha_k})\phi^i_{kj}(fe_{-\alpha_k}) \tag{1.3.11}$$

$$\phi^i_{kj}(fe_{-\alpha_k})\phi^j_{j0}(g) = \epsilon_k \phi^j_{ki}(ge_{-\alpha_k})\phi^i_{i0}(f) \tag{1.3.12}$$

It is easy to see that $\phi^{\overline{j}}_{0j}(\overline{g}) \subseteq \phi^j_{j0}(g)^*$ and similarly that $\phi^{\overline{j}}_{ik}(\overline{g}e_{\alpha_k}) \subseteq \phi^j_{ki}(ge_{-\alpha_k})^*$. Granted this, we obtain by alleviating notations

$$x_{i0}y^*_{j0} = \sum_k \lambda_k y^*_{ki}x_{kj} \tag{1.3.13}$$

$$x_{kj}y_{j0} = \epsilon_k y_{ki}x_{i0} \tag{1.3.14}$$

where the x_{qp}, y_{qp} commute with $L_{I^c}G$ and L_IG respectively. The above is an unbounded version of the transport formula (1.2.6) since $x_{i0}y^*_{j0}y_{j0} = \lambda_k\epsilon_k y^*_{ki}y_{ki}x_{i0}$.

A key lemma due to Wassermann [**Wa2**] asserts that the x_{qp} and y_{qp} above may be replaced by *bounded* operators $x_{qp} \in \mathrm{Hom}_{L_{I^c}G}(\mathcal{H}_p, \mathcal{H}_q)$, $y_{qp} \in \mathrm{Hom}_{L_IG}(\mathcal{H}_p, \mathcal{H}_q)$ without altering the relations (1.3.13)–(1.3.14) in such a way that x_{i0} and y_{j0} are replaced by their phases [4]. These do not quite give the transport formula yet but a further alteration of the operators based on the fact that the $\pi_p(L_IG)''$ are type III factors yields unitary x_{i0} and y_{j0}. Moreover, each $\lambda_k\epsilon_k$ is necessarily non–negative and vanishes iff λ_k does since ϵ_k turns out to be a root of unity. Thus,

$$x_{i0}y^*_{j0}y_{j0} = \sum_k \lambda_k\epsilon_k y^*_{ki}y_{ki}x_{i0} \tag{1.3.15}$$

As pointed out in §1.2 however, $x_{i0}y^*_{j0}y_{j0} = x_{i0}\pi_i(y^*_{j0}y_{j0})$ and therefore simplifying by x_{i0} we find (1.2.6). Combining the above with the discussion of §1.2, we obtain

Proposition 1.3.1. *Let $\mathcal{H}_i, \mathcal{H}_j$ be irreducible positive energy representations at level ℓ with lowest energy subspaces V_i, V_j. Assume in addition that the irreducible summands of $V_i \otimes V_j$ have multiplicity 1 and that all primary fields with charge V_i extend to bounded operator–valued distributions. Then*

$$\mathcal{H}_i \boxtimes \mathcal{H}_j = \bigoplus_k \mathcal{H}_k \tag{1.3.16}$$

where the sum spans the positive energy representations at level ℓ whose lowest energy subspace V_k is contained in $V_i \otimes V_j$ and such that the corresponding braiding coefficient λ_k in (1.3.9) *does not vanish.*

1.4. Statement of main results.

We adopt a more graphical notation and label the simplest representations of SO_{2n} by the corresponding Young diagrams. Thus, we denote the defining representation of SO_{2n} by $V_{\square}$ and its second exterior and symmetric (traceless) powers by $V_{\substack{\square\\\square}}$ and $V_{\square\square}$ respectively. Moreover, if U is an irreducible representation of $G = \mathrm{Spin}_{2n}$ admissible at level ℓ, we denote by $\mathcal{H}_U$ the corresponding positive energy representation of $L\,\mathrm{Spin}_{2n}$ whose lowest energy subspace is U.

Theorem 1.4.1. *Let U be an irreducible representation of SO_{2n} admissible at level ℓ. Then,*

$$\mathcal{H}_{\square} \boxtimes \mathcal{H}_U = \bigoplus_{W \subset V_{\square} \otimes U} N^W_{\square U}\mathcal{H}_W \tag{1.4.1}$$

[4] At present, the lemma only applies when all primary fields with charge V_i (or V_j) define *bounded* operator–valued distributions. We shall however only be concerned with this case in the present dissertation.

where $N_{\square U}^{W}=1$ *is* W *is admissible at level* ℓ *and* 0 *otherwise.*

The proof of theorem 1.4.1 rests on the following important special case

PROPOSITION 1.4.2. *The following fusion rules hold at level* ℓ

$$\mathcal{H}_{\square}\boxtimes\mathcal{H}_{\square}=\mathcal{H}_{\square\square}\oplus\mathcal{H}_{\substack{\square\\\square}}\oplus\mathcal{H}_0 \tag{1.4.2}$$

if $\ell\geq 2$ *and*

$$\mathcal{H}_{\square}\boxtimes\mathcal{H}_{\square}=\mathcal{H}_0 \tag{1.4.3}$$

if $\ell=1$.

PROOF OF PROPOSITION 1.4.2. We use proposition 1.3.1 and the tensor product rule

$$V_{\square}\otimes V_{\square}=V_{\square\square}\oplus V_{\substack{\square\\\square}}\oplus V_0 \tag{1.4.4}$$

The primary fields with charge $V_{\square}$ are shown to extend to bounded operator–valued distributions in chapter VI. The proof relies on the simple observation that they are essentially Fermi fields. The relevant braiding coefficients are computed in chapter VIII by explicitly solving the corresponding Knizhnik–Zamolodchikov equations and shown to be non–zero. The $\ell=1$ fusion rule for $\mathcal{H}_{\square}\boxtimes\mathcal{H}_{\square}$ differs from those for $\ell\geq 2$ because $V_{\square\square}$ and $V_{\substack{\square\\\square}}$ are not admissible at level 1 $\diamond$

PROOF OF THEOREM 1.4.1. We begin by using proposition 1.3.1 to obtain an upper bound on $\mathcal{H}_{\square}\boxtimes\mathcal{H}_U$. Since $V_{\square}$ is a minimal representation of Spin_{2n}, all irreducible summands of $V_{\square}\otimes U$ have multiplicity one. Moreover, the primary fields with charge $V_{\square}$ extend to bounded operator–valued distributions and therefore (1.4.1) holds with the $N_{\square U}^{W}$ replaced by some $0\leq \widetilde{N}_{\square U}^{W}\leq N_{\square U}^{W}$ since part of the braiding coefficients involved in the computation of fusion might vanish. The matrix $N_{\square}$ whose entries are the $N_{\square U}^{W}$, where U and W are single–valued representations of SO_{2n} is non–negative and irreducible [5]. By the Perron–Frobenius theorem [**GM**], $N_{\square}$ has, up to a multiplicative constant, a unique eigenvector with non–negative entries. On the other hand, if 2ρ is the sum of the positive roots of Spin_{2n} and χ_U the character of the irreducible Spin_{2n}–module U, the 'dimensions' $\delta_U=\chi_U(\exp(\frac{2\pi i\rho}{\ell+2(n-1)}))\in\mathbb{R}_+$ given in [**Ka1**, §13.8] obey, by a simple computation

$$\sum_{W\subset V_{\square}\otimes U} N_{\square U}^{W}\delta_W=\delta_{\square}\delta_U \tag{1.4.5}$$

and it follows that the Perron–Frobenius eigenvalue of $N_{\square}$ is equal to $\delta_{\square}$.

A similar statement about the matrix $\widetilde{N}_{\square}$ may be obtained by using some results of Doplicher–Haag–Roberts on the algebraic theory of superselection sectors [**DHR1, DHR2**]. In essence, the fact that $\mathcal{H}_{\square}\boxtimes\mathcal{H}_{\square}$ contains the vacuum representation $\mathcal{H}_0$, by virtue of proposition 1.4.2 implies that, on the ring $\mathcal{R}_0$ generated by the irreducible summands of the iterated powers $\mathcal{H}_{\square}^{\boxtimes m}$, $m\in\mathbb{N}$ there exists a unique positive character or *quantum dimension* function d which is additive under direct sums, multiplicative under fusion and normalised by $d(\mathcal{H}_0)=1$. Applying d to (1.4.1) with $N_{\square U}^{W}$ replaced by $\widetilde{N}_{\square U}^{W}$ yields

$$\sum_{W\subset V_{\square}\otimes U}\widetilde{N}_{\square U}^{W}d(\mathcal{H}_W)=d(\mathcal{H}_{\square})d(\mathcal{H}_U) \tag{1.4.6}$$

whenever U is a single–valued representation of SO_{2n} which appears in $\mathcal{R}_0$ [6]. We do not know $\widetilde{N}_{\square}$ to be irreducible, nor is (1.4.6) a statement about $\widetilde{N}_{\square}$ since some $\mathcal{H}_U$ may not lie in $\mathcal{R}_0$ but if $M_{\square}$ is the matrix whose entries are $\widetilde{N}_{\square U}^{W}$ if $\mathcal{H}_U,\mathcal{H}_W\in\mathcal{R}_0$ and zero otherwise, we clearly have $M_{\square}\leq N_{\square}$ entry–wise and therefore, by Perron–Frobenius theory, $d(\mathcal{H}_{\square})\leq\delta_{\square}$ with equality only if $M_{\square}=N_{\square}$,

---

[5] This is the reason for demanding that U be a single–valued representation of SO_{2n} in theorem 1.4.1. Without this restriction, $N_{\square}$ would have two irreducible diagonal blocks, corresponding respectively to the single and two–valued representations of SO_{2n}, and zero off–diagonal blocks and would therefore fail to be irreducible.

[6] In principle these are all the single–valued representations but that is part of what we are trying to establish.

i.e. only if (1.4.1) holds. Finally, a computation based on (1.4.2) and paralleling one of Wenzl [**We3**] shows that $d(\mathcal{H}_\square) = \delta_\square$ and we have therefore established (1.4.1) $\diamond$

THEOREM 1.4.3. *The positive energy representations of $L\,\mathrm{Spin}_{2n}$ whose lowest energy subspace is a single–valued representation of SO_{2n} are closed under fusion.*

PROOF. This follows at once because they are, by theorem 1.4.1 exactly the representations that appear in the iterated fusion powers $\mathcal{H}_\square^{\boxtimes m}$ $\diamond$

The restriction to single–valued representations of SO_{2n} in theorems 1.4.1 and 1.4.3 is technical rather than conceptual and these results conjecturally hold for all positive energy representations of $L\,\mathrm{Spin}_{2n}$. At present, we can show this at *odd* level by resorting to the action of the centre $Z(\mathrm{Spin}_{2n})$ via conjugation by discontinuous loops. Denote by $z\mathcal{H}$ the representation obtained by conjugating $\mathcal{H}$ by $z \in Z(\mathrm{Spin}_{2n})$ as in §1.1. We need the following

LEMMA 1.4.4. *Let $\mathcal{H}$ be a positive energy representation of $L\,\mathrm{Spin}_{2n}$ and $\mathcal{H}_0$ the vacuum representation at the same level. Then, for any $z \in Z(\mathrm{Spin}_{2n})$,*

$$z\mathcal{H}_0 \boxtimes \mathcal{H} \cong z\mathcal{H} \cong \mathcal{H} \boxtimes z\mathcal{H}_0 \tag{1.4.7}$$

PROOF. We prove the first identity only, the second follows in a similar way. Let $\zeta \in C^\infty([0,2\pi], \mathrm{Spin}_{2n})$ be the lift of a loop in $\mathrm{Spin}_{2n}/Z(\mathrm{Spin}_{2n})$ with $\zeta(2\pi)\zeta(0)^{-1} = z$. By definition, the conjugate $z\mathcal{K}$ of any positive energy representation $(\pi, \mathcal{K})$ is given by the isomorphism class of the representation $\gamma \to \pi(\zeta^{-1}\gamma\zeta)$ of $L\,\mathrm{Spin}_{2n}$ on $\mathcal{K}$. Form now the intertwiner spaces

$$\mathfrak{X} = \mathrm{Hom}_{L_{I^c}\,\mathrm{Spin}_{2n}}(\mathcal{H}_0, z\mathcal{H}_0) \qquad\qquad \mathfrak{Y} = \mathrm{Hom}_{L_I\,\mathrm{Spin}_{2n}}(\mathcal{H}_0, \mathcal{H}) \tag{1.4.8}$$

Choosing a ζ equal to one on I so that $\pi_0(\zeta^{-1}\gamma\zeta) = \pi_0(\gamma)$ for any $\gamma \in L_{I^c}\,\mathrm{Spin}_{2n}$, we have

$$\mathfrak{X} = \mathrm{Hom}_{L_{I^c}\,\mathrm{Spin}_{2n}}(\mathcal{H}_0, \mathcal{H}_0) = \pi_0(L_I\,\mathrm{Spin}_{2n})'' \tag{1.4.9}$$

where the last identity follows by Haag duality. We claim now that the map $U : \mathfrak{X} \otimes \mathfrak{Y} \to \mathcal{H}$, $x \otimes y \to yx\Omega$ is norm–preserving. Indeed,

$$\|x \otimes y\|^2 = (x^*xy^*y\Omega, \Omega) = (x^*y^*yx\Omega, \Omega) = \|yx\Omega\|^2 \tag{1.4.10}$$

where we have used locality so that $y^*y \in \mathrm{Hom}_{L_I\,\mathrm{Spin}_{2n}}(\mathcal{H}_0, \mathcal{H}_0)$ commutes with x. U is equivariant for the actions of $L_I\,\mathrm{Spin}_{2n} \times L_{I^c}\,\mathrm{Spin}_{2n}$ on $\mathfrak{X} \otimes \mathfrak{Y}$ and $z\mathcal{H}$ and extends to an isometry $z\mathcal{H}_0 \boxtimes \mathcal{H} \to z\mathcal{H}$ whose range is dense by the Reeh–Schlieder theorem. Thus, U is the required unitary equivalence $\diamond$

THEOREM 1.4.5.

(i) *Let $\mathcal{H}_U$ be the irreducible positive energy representations of $L\,\mathrm{Spin}_{2n}$ at odd level ℓ whose lowest energy subspace is U. Then*

$$\mathcal{H}_\square \boxtimes \mathcal{H}_U = \bigoplus_{W \subset V_\square \otimes U} N_{\square U}^W \mathcal{H}_W \tag{1.4.11}$$

where $N_{\square U}^W$ is 1 if W is admissible at level ℓ and zero otherwise.

(ii) *The positive energy representations of $L\,\mathrm{Spin}_{2n}$ at odd level ℓ are closed under fusion.*

PROOF. (i) By theorem 1.4.1, (1.4.11) holds if U is a single–valued representation of SO_{2n}. If that is the case, we find by conjugating both sides by an element $z \in Z(\mathrm{Spin}_{2n})$ and using lemma 1.4.4 that

$$\mathcal{H}_\square \boxtimes z\mathcal{H}_U = \bigoplus_{W \subset V_\square \otimes U} N_{\square U}^W \, z\mathcal{H}_W \tag{1.4.12}$$

By definition, $z\mathcal{H}_U = \mathcal{H}_{zU}$ and $z\mathcal{H}_W = \mathcal{H}_{zW}$ where the notation refers to the induced action of $Z(\mathrm{Spin}_{2n})$ on the irreducible Spin_{2n}–modules admissible at level ℓ. The explicit form of this action is computed in chapter I and shows in particular that $W \subset U \otimes V_\square$ if, and only if $zW \subset V_\square \otimes zU$. Thus,

$$\mathcal{H}_\square \boxtimes \mathcal{H}_{zU} = \bigoplus_{W' \subset V_\square \otimes zU} N_{\square zU}^{W'} \, \mathcal{H}_{W'} \tag{1.4.13}$$

Moreover, when ℓ is odd, any two–valued representation U' of SO_{2n} may be written as zU where $z \in Z(\mathrm{Spin}_{2n})$ and U is a single–valued representation of SO_{2n}. This yields (i).

(ii) By theorem 1.4.3, $\mathcal{H}_{U_1} \boxtimes \mathcal{H}_{U_2}$ is of positive energy if U_1 and U_2 are single–valued SO_{2n}–modules. Conjugating successively by z_2 and $z_1 \in Z(\mathrm{Spin}_{2n})$ and using lemma 1.4.4, we find that

$$\mathcal{H}_{z_1 U_1} \boxtimes \mathcal{H}_{z_2 U_2} \cong z_1 \mathcal{H}_0 \boxtimes (\mathcal{H}_{U_1} \boxtimes \mathcal{H}_{U_2}) \boxtimes z_2 \mathcal{H}_0 \cong z_1 z_2 (\mathcal{H}_{U_1} \boxtimes \mathcal{H}_{U_2}) \tag{1.4.14}$$

is of positive energy. Since any irreducible Spin_{2n}–module admissible at odd level ℓ may be written as zU for some $z \in Z(\mathrm{Spin}_{2n})$ and U a single–valued SO_{2n}–module, the result follows $\diamond$

THEOREM 1.4.6. *The level 1 representations of* $L\,\mathrm{Spin}_{2n}$ *are closed under fusion. Moreover, if* $\mathcal{H}_0$ *is the level 1 vacuum representation, then*

$$z \longrightarrow z\mathcal{H}_0 \tag{1.4.15}$$

yields an isomorphism of the group algebra $\mathbb{C}[Z(\mathrm{Spin}_{2n})]$ *and the level 1 fusion ring of* $L\,\mathrm{Spin}_{2n}$.

PROOF. The above map is bijective since, as noted in §1.1, the action of $Z(\mathrm{Spin}_{2n})$ on the level 1 representations is transitive and free. Moreover, by lemma 1.4.4,

$$z_1 \mathcal{H}_0 \boxtimes z_2 \mathcal{H}_0 \cong z_1 (z_2 \mathcal{H}_0) = z_1 z_2 \mathcal{H}_0 \tag{1.4.16}$$

$\diamond$

Part 1

Positive energy representations and primary fields

CHAPTER I

Positive energy representations and primary fields

This chapter introduces the basic objects of study of this dissertation, the *positive energy representations* of the loop group $LG = C^\infty(S^1, G)$ of the Lie group $G = \mathrm{Spin}_{2n}$, $n \geq 3$. These are projective unitary representations supporting an intertwining action of $\mathrm{Rot}\, S^1$ with finite–dimensional eigenspaces and positive spectrum. As explained in section 1, they are completely reducible and the irreducible ones are classified by their *level* $\ell \in \mathbb{N}$, which determines the underlying projective cocycle, and their *lowest energy subspace*, an irreducible G–module. Moreover, only finitely many irreducibles exist at any given level ℓ. In section 2, we restrict our attention to the level 1 representations of LG and their lowest energy subspaces, the *minimal* G–modules. These correspond in a natural way to elements in the dual of the centre of G and play an important rôle in view of the fact that any level ℓ representation is contained in the ℓ–fold tensor product $\mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_\ell}$ of level 1 representations.

In section 3, we show that the outer action of the centre $Z(G)$ on LG via conjugation by discontinuous loops, *i.e.* lifts to G of loops in $G/Z(G)$, induces a level preserving one on the positive energy representations of LG. At level 1, the action is transitive and free, a fact which will be used repeatedly, most notably in chapter IX to prove the isomorphism of the level 1 fusion ring of LG and the group algebra of $Z(G)$. Finally, in section 4, we outline the classification of the *primary fields* of LG. These are algebraic, operator–valued distributions mapping between positive energy representations of equal level. The study of their analytic and algebraic properties, both of which are essential ingredients in the computation of fusion, will be carried out in chapters V–VI and VII–VIII respectively.

1. Definition and classification of positive energy representations

In this section and the rest of this chapter, G denotes a compact, connected and simply–connected simple Lie group with Lie algebra $\mathfrak{g}$.

1.1. The loop group LG.

The *loop group* of G is, by definition $LG = C^\infty(S^1, G)$. When endowed with the C^∞–topology and pointwise multiplication, LG is a real analytic Fréchet Lie group. Its Lie algebra is $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ with pointwise bracket. It has complexification $L\mathfrak{g}_\mathbb{C} = C^\infty(S^1, \mathfrak{g}_\mathbb{C})$ and is often more conveniently regarded as the -1 eigenspace of the anti–linear anti–involution $X \to -\overline{X}$ determined by the canonical conjugation on $\mathfrak{g}_\mathbb{C}$. As Lie groups, $LG \cong \Omega G \rtimes G$ where ΩG is the space of based loops on which G acts by conjugation and the map is given by $\gamma \to (\gamma\gamma(0)^{-1}, \gamma(0))$. Since $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$, LG is connected and simply–connected. In particular, it is generated by the image of the exponential map since the latter is locally one–to–one.

LG admits a smooth automorphic action of $\mathrm{Rot}\, S^1$ given by $R_\theta\gamma = \gamma_\theta$ where $\gamma_\theta(\phi) = \gamma(\phi - \theta)$. The corresponding semi-direct product $LG \rtimes \mathrm{Rot}\, S^1$ is therefore a Fréchet Lie group, which however fails to be analytic, since for a fixed $\gamma \in LG$, $\theta \to \gamma_\theta$ is analytic iff γ is an analytic loop in G. Identifying the Lie algebra of $\mathrm{Rot}\, S^1$ and $i\mathbb{R}$ with generator id so that $R_\theta = \exp(i\theta d)$, the Lie algebra of $LG \rtimes \mathrm{Rot}\, S^1$ is $L\mathfrak{g} \rtimes i\mathbb{R}$ and the following relations hold for $\gamma \in LG$ and $X \in L\mathfrak{g}$

$$\mathrm{Ad}(R_\theta)X = X_\theta \tag{1.1.1}$$

$$[id, X] = \left.\frac{d}{d\theta}\right|_{\theta=0} \mathrm{Ad}(R_\theta)X = -\dot{X} \tag{1.1.2}$$

and

$$\mathrm{Ad}(\gamma)id = \frac{d}{d\theta}\bigg|_{\theta=0} \gamma R_\theta \gamma^{-1} = id - \gamma(\dot{\gamma^{-1}}) = id + \dot{\gamma}\gamma^{-1} \tag{1.1.3}$$

where we have used $0 = (\dot{\gamma\gamma^{-1}}) = \dot{\gamma}\gamma^{-1} + \gamma(\dot{\gamma^{-1}})$.

We shall be concerned with projective representations of $LG \rtimes \mathrm{Rot}\, S^1$ and will therefore need to consider continuous Lie algebra cocycles on $L\mathfrak{g} \rtimes i\mathbb{R}$, *i.e.* skew–symmetric, $i\mathbb{R}$–valued maps β satisfying the Jacobi identity

$$\beta([X,Y],Z) + \beta([Z,X],Y) + \beta([Y,Z],X) = 0 \tag{1.1.4}$$

Up to the addition of coboundaries $\beta(X,Y) = d\alpha(X,Y) = \alpha([X,Y])$, these vanish on the Lie algebra of $\mathrm{Rot}\, S^1$ and their restriction to $L\mathfrak{g}$ is a multiple of the fundamental cocycle [**PS**, 4.2.4]

$$iB(X,Y) = i\int_0^{2\pi} \langle X, \dot{Y}\rangle \frac{d\theta}{2\pi} \tag{1.1.5}$$

where $\langle\cdot,\cdot\rangle$ is the *basic inner product* on $\mathfrak{g}_{\mathbb{C}}$, *i.e.* the unique multiple of the Killing form for which the highest root θ has squared length 2.

$L\mathfrak{g}$ contains a distinguished dense sub–algebra $L^{\mathrm{pol}}\mathfrak{g}$ consisting of all $\mathfrak{g}$–valued trigonometric polynomials. Its complexification $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ is spanned by the elements $x(n) = x \otimes e^{in\theta}$, $x \in \mathfrak{g}_{\mathbb{C}}$, $n \in \mathbb{Z}$ with bracket $[x(n), y(m)] = [x,y](n+m)$. The restriction of the cocycle (1.1.5) to $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ reads

$$iB(x(n), y(m)) = n\delta_{n+m,0}\langle x, y\rangle \tag{1.1.6}$$

The action of $\mathrm{Rot}\, S^1$ leaves $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ invariant and satisfies $[d, x(n)] = -nx(n)$. The central extension of $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ with central term c and cocycle given by (1.1.6) is usually denoted by $\widetilde{\mathfrak{g}_{\mathbb{C}}}$. The semi–direct product $\widetilde{\mathfrak{g}_{\mathbb{C}}} \rtimes \mathbb{C}d$ is the affine, untwisted Kac–Moody algebra $\widehat{\mathfrak{g}_{\mathbb{C}}}$ corresponding to $\mathfrak{g}_{\mathbb{C}}$ [**Ka1**][1]. $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ possesses two decompositions we shall make use of. The first is the triangular decomposition determined by a maximal torus $T \subset G$

$$L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\leq} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\geq} \tag{1.1.7}$$

where $\mathfrak{t}_{\mathbb{C}}$ is the complexified Lie algebra of T and $\mathfrak{g}_{\leq}$ (resp. $\mathfrak{g}_{\geq}$) is the nilpotent Lie algebra spanned by the $x(n)$ with $n < 0$ (resp. $n > 0$) and $x \in \mathfrak{g}_{\mathbb{C}}$ or $n = 0$ and x lying in a negative (resp. positive) root space of $\mathfrak{g}_{\mathbb{C}}$. The second decomposition is given by

$$L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{-} \oplus \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{+} \tag{1.1.8}$$

where $\mathfrak{g}_{\pm}$ are spanned by the $x(n)$, $x \in \mathfrak{g}_{\mathbb{C}}$, $n \gtrless 0$.

1.2. Positive energy representations of LG.

We outline below the classification of positive energy representations of LG following [**Wa2**] to which we refer for further details. The basic properties of projective representations and central extensions relevant to the present discussion may be found in §1.3. Let π be a projective unitary representation of $LG \rtimes \mathrm{Rot}\, S^1$ on a complex Hilbert space $\mathcal{H}$, *i.e.* a strongly continuous homomorphism

$$\pi : LG \rtimes \mathrm{Rot}\, S^1 \longrightarrow PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T} \tag{1.2.1}$$

Over $\mathrm{Rot}\, S^1$, π possesses a continuous lift to a unitary representation which we denote by the same symbol. By definition, $\mathcal{H}$ is of *positive energy* if $\mathcal{H} = \bigoplus_{n \geq n_0} \mathcal{H}(n)$ with $\dim \mathcal{H}(n) < \infty$ and $\pi(R_\theta)|_{\mathcal{H}(n)} = e^{in\theta}$. The lift is clearly only determined up to multiplication by a character of $\mathrm{Rot}\, S^1$ and we normalise it by choosing $n_0 = 0$ and $\mathcal{H}(0) \neq 0$. Since G is simple, connected and simply–connected, the restriction of π to G lifts uniquely to a strongly continuous unitary representation which we also denote by π. It commutes with the unitary action of $\mathrm{Rot}\, S^1$ since, projectively $\pi(g)\pi(R_\theta)\pi(g)^*\pi(R_\theta)^* = 1$. Thus, lifting each representation, the following holds in $U(\mathcal{H})$

$$\pi(g)\pi(R_\theta)\pi(g)^*\pi(R_\theta)^* = \chi(g,\theta) \tag{1.2.2}$$

[1] The d above is the opposite of that used in the theory of affine Kac–Moody algebras.

where $\chi(g,\theta) \in \mathbb{T}$ is continuous and multiplicative in either variable and therefore defines a continuous homomorphism $G \to \mathrm{Hom}(\mathrm{Rot}\, S^1, \mathbb{T}) \cong \mathbb{Z}$. Since G is connected, $\chi \equiv 1$ as claimed and it follows in particular that each $\mathcal{H}(n)$ is a finite–dimensional G–module.

The spectral assumption on the action of $\mathrm{Rot}\, S^1$ implies that a positive energy representation is completely reducible. Since it is projective however, some care is needed in defining the direct sum of its irreducible components. Indeed, as explained in §1.3, the direct sum of two projective representations $(\pi_i, \mathcal{H}_i)$, $i = 1, 2$ of a topological group Γ may be defined only if the corresponding central extensions

$$\pi_i^* U(\mathcal{H}_i) = \{(g, u) \in \Gamma \times U(\mathcal{H}_i) | \pi_i(g) = p_i(u)\} \tag{1.2.3}$$

obtained by pulling back to Γ the canonical central extensions

$$1 \to \mathbb{T} \to U(\mathcal{H}_i) \xrightarrow{p_i} PU(\mathcal{H}_i) \tag{1.2.4}$$

are isomorphic. When that is the case, the definition of $\pi_1 \oplus \pi_2$ depends upon the choice of an identification $\pi_1^* U(\mathcal{H}_1) \cong \pi_2^* U(\mathcal{H}_2)$ which is unique only up to multiplication by an element of $\mathrm{Hom}(\Gamma, \mathbb{T})$.

Since $\mathrm{Hom}(LG, \mathbb{T}) = \{1\}$ [**PS**, prop. 3.4.3] and $\mathrm{Hom}(LG \rtimes \mathrm{Rot}\, S^1, \mathbb{T}) = \mathbb{Z}$, the direct sum of projective representations of $LG \rtimes \mathrm{Rot}\, S^1$ is ill–defined and we shall therefore forgetfully regard positive energy representations as LG–modules. This does not affect the irreducible ones since a positive energy representation is irreducible under $LG \rtimes \mathrm{Rot}\, S^1$ if, and only if it remains so when restricted to LG [**PS**, prop. 9.2.3] and leads to a canonically, if partially defined notion of direct sum. Thus, we regard two positive energy representations $(\pi_i, \mathcal{H}_i)$ as unitarily equivalent if they are so as projective LG–modules, *i.e.* if there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(\gamma)U^* = \pi_2(\gamma)$ in $PU(\mathcal{H}_2)$ for all $\gamma \in LG$. If both are irreducible, this is easily seen to imply their unitary equivalence as projective $LG \rtimes \mathrm{Rot}\, S^1$–modules.

The classification of a positive energy representation $(\pi, \mathcal{H})$ is obtained via the associated infinitesimal action of $L^{\mathrm{pol}}\mathfrak{g}$ in the following way. Consider the subspace $\mathcal{H}^{\mathrm{fin}} \subset \mathcal{H}$ of *finite energy vectors* for $\mathrm{Rot}\, S^1$, that is the algebraic direct sum $\sum_{n \geq 0} \mathcal{H}(n)$. The latter is a core for the normalised self–adjoint generator of rotations which we abusively denote by d. Thus

$$d|_{\mathcal{H}(n)} = n \qquad \text{and} \qquad \pi(\exp_{\mathrm{Rot}\, S^1}(i\theta d)) = e^{i\theta d} \tag{1.2.5}$$

For any $X \in L^{\mathrm{pol}}\mathfrak{g}$, the one–parameter projective group $\pi(\exp_{LG}(tX))$ possesses a continuous lift to $U(\mathcal{H})$, unique up to multiplication by a character of $\mathbb{R}$. It is therefore given, via Stone's theorem by $e^{t\pi(X)}$ where $\pi(X)$ is a skew–adjoint operator determined up to an additive constant.

THEOREM 1.2.1. *The subspace $\mathcal{H}^{\mathrm{fin}}$ of finite energy vectors is an invariant core for the operators $\pi(X)$, $X \in L^{\mathrm{pol}}\mathfrak{g}$. The operators $\pi(X)$ may be chosen uniquely so as to satisfy $[d, \pi(X)] = i\pi(\dot{X})$ on $\mathcal{H}^{\mathrm{fin}}$ and then $X \to \pi(X)$ gives a projective representation of $L^{\mathrm{pol}}\mathfrak{g}$ on $\mathcal{H}^{\mathrm{fin}}$ such that*

$$[\pi(X), \pi(Y)] = \pi([X, Y]) + i\ell B(X, Y) \tag{1.2.6}$$

where $iB(X, Y)$ is given by (1.1.5) *and ℓ is a non–negative integer called the level of the representation.*

When no confusion arises, we denote the restriction of the operators $\pi(X)$, $X \in L^{\mathrm{pol}}\mathfrak{g}$ to $\mathcal{H}^{\mathrm{fin}}$ by the same symbol and extend the resulting projective representation $\pi : L^{\mathrm{pol}}\mathfrak{g} \to \mathrm{End}(\mathcal{H}^{\mathrm{fin}})$ to one of $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ satisfying (1.2.6) as well as the formal adjunction property $\pi(X)^* = -\pi(\overline{X})$.

By theorem 1.2.1, the operators $\pi(X)$ and d give rise to a unitarisable representation of the Kac–Moody algebra $\widehat{\mathfrak{g}_{\mathbb{C}}}$ at level ℓ such that d is diagonal with finite–dimensional eigenspaces and spectrum in $\mathbb{N}$. Such representations split into a direct sum of irreducibles, each of which is necessarily an integrable highest weight representation, that is a module generated over the enveloping algebra $\mathfrak{U}(\mathfrak{g}_{<})$ by a vector v annihilated by $\mathfrak{g}_{\geq}$ and diagonalising the action of $T \rtimes \mathrm{Rot}\, S^1$. Thus, for any $h \in \mathfrak{t}_{\mathbb{C}}$

$$dv = nv \tag{1.2.7}$$

$$\pi(h)v = \lambda(h)v \tag{1.2.8}$$

for some $n \in \mathbb{N}$ and dominant integral weight λ of G satisfying $\langle\lambda,\theta\rangle \leq \ell$ where θ is the highest root. The pair (ℓ,λ) classifies the representation as an $L^{\mathrm{pol}}\mathfrak{g}$–module uniquely [**Ka1**]. The highest weight representation is also generated over the enveloping algebra $\mathfrak{U}(\mathfrak{g}_-)$ by its *lowest energy subspace*, *i.e.* the d–eigenspace with lowest eigenvalue which in fact coincides with the irreducible G–module with highest weight λ generated by v. We shall interchangeably adopt either point of view. When the level is understood, we refer to λ as the highest weight of the representation. The collection of dominant integral weights of G satisfying $\langle\lambda,\theta\rangle \leq \ell$ is a finite set called the level ℓ *alcove* and is denoted by $\mathcal{A}_\ell$.

Propositions 1.2.2 and 1.2.3 below imply that the classification of positive energy representations of LG is equivalent to that of their finite energy subspaces as $L^{\mathrm{pol}}\mathfrak{g}$–modules.

Proposition 1.2.2. *Let $(\pi,\mathcal{H})$ be a positive energy representation of LG with finite energy subspace $\mathcal{H}^{\mathrm{fin}}$. If $\mathcal{K}\subset\mathcal{H}^{\mathrm{fin}}$ is invariant under $L^{\mathrm{pol}}\mathfrak{g}\rtimes\mathbb{C}d$, then $\overline{\mathcal{K}}$ is invariant under $LG\rtimes\mathrm{Rot}\,S^1$. In particular, $\mathcal{H}$ is topologically irreducible under $LG\rtimes\mathrm{Rot}\,S^1$ if, and only if $\mathcal{H}^{\mathrm{fin}}$ is algebraically irreducible for $L^{\mathrm{pol}}\mathfrak{g}\rtimes\mathbb{C}d$.*

Proof. By continuity of π, it is sufficient to prove the invariance of $\overline{\mathcal{K}}$ under the dense subgroup generated by $\exp_{LG}(L^{\mathrm{pol}}\mathfrak{g})$. Let $X\in L^{\mathrm{pol}}\mathfrak{g}$ and $\pi(X)$ the corresponding skew–adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(\pi(X))$ and invariant core $\mathcal{H}^{\mathrm{fin}}\subset\mathcal{D}(\pi(X))$. It is sufficient to prove that $P\pi(X)\subset\pi(X)P$ where P is the orthogonal projection on $\overline{\mathcal{K}}$ for then, by the spectral theorem, P commutes with bounded functions of $\pi(X)$ and in particular with $e^{t\pi(X)}=\pi(\exp_{LG}(tX))$. Notice that $\mathcal{K}$ is invariant under d and is therefore a graded subspace of $\mathcal{H}^{\mathrm{fin}}$ so that it coincides with the finite energy subspace of $\overline{\mathcal{K}}$. Since P commutes with $\mathrm{Rot}\,S^1$, it follows that its restriction to $\mathcal{H}^{\mathrm{fin}}$ is the orthogonal projection onto $\mathcal{K}$ and therefore $\pi(X)P\eta = P\pi(X)\eta$ for any $\eta\in\mathcal{H}^{\mathrm{fin}}$. Let now $\xi\in\mathcal{D}(\pi(X))$ and $\xi_n\in\mathcal{H}^{\mathrm{fin}}$ a sequence such that $\xi_n\to\xi$ and $\pi(X)\xi_n\to\pi(X)\xi$. Then $\eta_n=P\xi_n\in\mathcal{H}^{\mathrm{fin}}\subset\mathcal{D}(\pi(X))$ converges to $P\xi$ and $\pi(X)\eta_n = P\pi(X)\xi_n\to P\pi(X)\xi$ whence $P\pi(X)\subset\pi(X)P$ as claimed. To conclude, notice that $\overline{\mathcal{K}}=\mathcal{H}$ iff $\mathcal{K}^{\mathrm{fin}}=\mathcal{H}^{\mathrm{fin}}$ and therefore topological irreducibility of $\mathcal{H}$ implies algebraic irreducibility of $\mathcal{H}^{\mathrm{fin}}$. The converse holds by complete reducibility of positive energy representations and functoriality $\diamond$

Proposition 1.2.3. *Two positive energy representations $\mathcal{H}_1$ and $\mathcal{H}_2$ are unitarily equivalent as LG–modules if, and only if their finite energy subspaces are isomorphic as $L^{\mathrm{pol}}\mathfrak{g}$–modules.*

Evidently, the unitarily equivalence of $\mathcal{H}_1,\mathcal{H}_2$ as LG–modules implies that of $\mathcal{H}_1^{\mathrm{fin}},\mathcal{H}_2^{\mathrm{fin}}$ as $L^{\mathrm{pol}}\mathfrak{g}$–modules. By complete reducibility, we need only prove the converse when the $\mathcal{H}_i$ and therefore the $\mathcal{H}_i^{\mathrm{fin}}$ are irreducible. Let $U:\mathcal{H}_1^{\mathrm{fin}}\to\mathcal{H}_2^{\mathrm{fin}}$ be an isomorphism of $L^{\mathrm{pol}}\mathfrak{g}$–modules. Up to multiplication by a scalar, U is an isometry since highest weight representations admit a unique $L^{\mathrm{pol}}\mathfrak{g}$–invariant inner product. Thus U extends to a unitary $\mathcal{H}_1\to\mathcal{H}_2$ such that $U\pi_1(X)U^*=\pi_2(X)$ on $\mathcal{H}_2^{\mathrm{fin}}$ for any $X\in L^{\mathrm{pol}}\mathfrak{g}$. Since $\mathcal{H}_i^{\mathrm{fin}}$ is a core for $\pi_i(X)$, it follows that $U\pi_1(X)U^*=\pi_2(X)$ holds as an operator identity and therefore that U intertwines with the one parameter groups $\exp_{LG}(tX)$, $X\in L^{\mathrm{pol}}\mathfrak{g}$. Since these generate a dense subgroup in LG, U is an LG intertwiner $\diamond$

Corollary 1.2.4. *An irreducible, positive energy representation of LG is uniquely determined by its level $\ell\in\mathbb{N}$ and its lowest energy subspace, an irreducible G–module whose highest weight λ satisfies $\langle\lambda,\theta\rangle\leq\ell$.*

Definition. An irreducible G–module V whose highest weight λ satisfies $\langle\lambda,\theta\rangle\leq\ell$ will be called *admissible at level ℓ*. The corresponding positive energy representation will be denoted by $\mathcal{H}_V$ or $\mathcal{H}_\lambda$.

Remark. We will show in chapter II (proposition 2.4.3) that the central extensions of LG corresponding to positive energy representations of levels ℓ_1,ℓ_2 are isomorphic if, and only if $\ell_1=\ell_2$. In particular, the direct sum of two positive energy representations of equal level is unambiguously defined.

Corollary 1.2.4 settles the uniqueness of irreducible positive energy representations of LG corresponding to a given level ℓ and irreducible G–module V. The corresponding existence problem may be solved by one of the following methods

(i) It is easy to construct the corresponding highest weight representation $\mathcal{V}$ of $\widehat{\mathfrak{g}_{\mathbb{C}}}$ using Verma modules and to prove that $\mathcal{V}$ is unitarisable [**Ka1**]. Some simple estimates of Goodman and Wallach [**GoWa**] then show that the action of $\widetilde{\mathfrak{g}_{\mathbb{C}}}$ extends to one of the Lie algebra $L\mathfrak{g}_{\mathbb{C}}$ on a suitable Fréchet completion of $\mathcal{V}$. This action may then be exponentiated to a projective unitary representation of LG on the Hilbert space completion of $\mathcal{V}$. This is carried out for the dense subgroup of analytic loops $S^1 \to G$ in [**GoWa**] and, by different methods for the full loop group LG in [**TL1**].

(ii) One may alternatively consider the space of holomorphic section of a suitable vector bundle with fibre V over the flag manifold LG/G. This space possesses an LG–invariant inner product and its Hilbert space completion is $\mathcal{H}_V$ [**PS**, chap. 11].

The positive energy representations of the loop groups relevant to this thesis, namely those corresponding to $G = \mathrm{Spin}_{2n}$, $n \geq 3$ possess an alternative Fermionic construction which will be given in chapter III.

1.3. Appendix : projective representations and central extensions.

We give the definitions and elementary properties of central extensions and projective representations and discuss in particular the problems associated with defining the direct sum of the latter. All groups below are assumed to be topological and the associated homomorphisms continuous.

Definition. Let Γ, A be groups with A abelian. A *central extension* of Γ by A is a short exact sequence

$$1 \to A \xrightarrow{i} \widetilde{\Gamma} \xrightarrow{p} \Gamma \to 1 \tag{1.3.1}$$

with $i(A) < Z(\widetilde{\Gamma})$. A homomorphism of central extensions is a map $\phi : \widetilde{\Gamma}_1 \to \widetilde{\Gamma}_2$ making the following a commutative diagram

$$\begin{tikzcd} & & \widetilde{\Gamma}_1 \arrow[dd, "\phi"] \arrow[dr] & & \\ 1 \arrow[r] & A \arrow[ur] \arrow[dr] & & \Gamma \arrow[r] & 1 \\ & & \widetilde{\Gamma}_2 \arrow[ur] & & \end{tikzcd} \tag{1.3.2}$$

Given two central extensions $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$ we may form their *product* $\widetilde{\Gamma} = \widetilde{\Gamma}_1 \star \widetilde{\Gamma}_2$ as the quotient of

$$\{(\widetilde{g}_1, \widetilde{g}_2) \in \widetilde{\Gamma}_1 \times \widetilde{\Gamma}_2 |\, p_1(\widetilde{g}_1) = p_2(\widetilde{g}_2)\} \tag{1.3.3}$$

by the image of the diagonal embedding $A \to \widetilde{\Gamma}_1 \times \widetilde{\Gamma}_2$, $a \to (i_1(a), i_2(a^{-1}))$. The associated maps are given by $p(\widetilde{g}_1, \widetilde{g}_2) = p_1(\widetilde{g}_1) = p_2(\widetilde{g}_2)$ and $i(a) = (i_1(a), 1) = (1, i_2(a))$. The set of isomorphism classes of central extensions of Γ by A endowed with this product operation is easily seen to be an abelian group, usually denoted by $H^2(\Gamma, A)$ with the trivial central extension $\Gamma \times A$ as identity element. The inverse $\overline{\widetilde{\Gamma}}$ of the central extension (1.3.1) is given by $\overline{\widetilde{\Gamma}} = \widetilde{\Gamma}$, $\overline{p} = p$, $\overline{\imath}(a) = i(a^{-1})$.

Let $\mathcal{H}$ be a complex Hilbert space. We endow the unitary group $U(\mathcal{H})$ with the strong operator topology and the projective unitary group $PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}$ with the corresponding quotient topology.

Definition. A projective unitary representation of Γ on $\mathcal{H}$ is a continuous homomorphism $\pi : \Gamma \to PU(\mathcal{H})$. Two such representations $(\mathcal{H}_i, \pi_i)$, $i = 1, 2$ are *unitarily equivalent* if there exists a unitary $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that for any $g \in \Gamma$,

$$V\pi_1(g)V^* = \pi_2(g) \text{ in } PU(\mathcal{H}_2) \tag{1.3.4}$$

A projective unitary representation is *irreducible* if it leaves no proper, non–trivial subspace of $\mathcal{H}$ invariant or equivalently if $T \in B(\mathcal{H})$ commutes with π iff T is a scalar.

Remark.

(i) Two inequivalent unitary representations of a group Γ may become equivalent when regarded as projective representations. For example, all characters of $\mathbb{T} = U(1)$ are projectively equivalent. However, this is the case iff the original representations differ by a character of Γ.

(ii) The irreducibility of projective representations is a well–defined concept since the map $U(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, $(V, T) \rightarrow VTV^*$ descends to $PU(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ and it therefore makes sense for an operator, and in particular a projection to commute with $\pi(g) \in PU(\mathcal{H})$ for any $g \in \Gamma$.

By keeping track of the phases of the operators $\pi(g)$, a projective unitary representation $(\mathcal{H}, \pi)$ may be viewed as a unitary representation of the group

$$\pi^* U(\mathcal{H}) = \{(g, V) \in \Gamma \times U(\mathcal{H}) | \, \pi(g) = [V]\} \tag{1.3.5}$$

where $[V]$ is the equivalence class of V in $PU(\mathcal{H})$, by defining $\widetilde{\pi}(g, V) = V$. $\pi^* U(\mathcal{H})$ is in fact the central extension of Γ by $\mathbb{T}$ obtained by pulling back the canonical central extension

$$1 \rightarrow \mathbb{T} \rightarrow U(\mathcal{H}) \rightarrow PU(\mathcal{H}) \rightarrow 1 \tag{1.3.6}$$

The isomorphism class of $\pi^* U(\mathcal{H})$ depends on the particular representation and it is easy to see that $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are unitarily equivalent iff the corresponding central extensions are isomorphic and the Hilbert spaces are unitarily equivalent as representations of either of the extensions.

The homomorphism $U(\mathcal{H}_1) \times U(\mathcal{H}_2) \rightarrow U(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $(V_1, V_2) \rightarrow V_1 \otimes V_2$ descends to the corresponding projective quotients, and we may therefore define the tensor product $\pi_1 \otimes \pi_2$ of two projective unitary representations π_1, π_2. The map $((g, V_1), (g, V_2)) \rightarrow (g, V_1 \otimes V_2)$ then yields an isomorphism

$$\pi_1^* U(\mathcal{H}_1) \star \pi_2^* U(\mathcal{H}_2) \cong (\pi_1 \otimes \pi_2)^* U(\mathcal{H}_1 \otimes \mathcal{H}_2) \tag{1.3.7}$$

Similarly, the map $U(\mathcal{H}) \rightarrow U(\overline{\mathcal{H}})$, $V \rightarrow IVI^{-1}$ where $\overline{\mathcal{H}}$ is $\mathcal{H}$ endowed with the opposite complex structure and $I : \mathcal{H} \rightarrow \overline{\mathcal{H}}$ is the canonical anti–linear identification, descends to the projective quotients. We may therefore define the conjugate $\overline{\pi}$ of a projective representation π and the map $(g, V) \rightarrow (g, IVI^{-1})$ gives an isomorphism

$$\overline{\pi^* U(\mathcal{H})} \cong \overline{\pi}^* U(\overline{\mathcal{H}}) \tag{1.3.8}$$

On the other hand, the map

$$U(\mathcal{H}_1) \times U(\mathcal{H}_2) \rightarrow U(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad (V_1, V_2) \rightarrow \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \tag{1.3.9}$$

does **not** descend to the projective quotients and it is therefore impossible in general to define the direct sum of projective unitary representations. In fact, if $(\mathcal{H}_i, \pi_i)$ are two projective representations of Γ possessing a direct sum, *i.e.* a projective representation π on $\mathcal{H}_1 \oplus \mathcal{H}_2$ leaving the summands invariant and restricting on each $\mathcal{H}_i$ to π_i, then $\pi_1^* U(\mathcal{H}_1) \cong \pi_2^* U(\mathcal{H}_2)$. The isomorphism is explicitly given by $(g, V_1) \rightarrow (g, V_2)$ where $V_2 \in U(\mathcal{H}_2)$ is uniquely determined by

$$[V_2] = \pi_2(g) \qquad \text{and} \qquad \left[\begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}\right] = \pi(g) \tag{1.3.10}$$

When $\pi_1^* U(\mathcal{H}_1) \cong \pi_2^* U(\mathcal{H}_2)$, *a* direct sum may be defined as the projectivisation of the direct sum representation $\widetilde{\pi}_1 \oplus \widetilde{\pi}_2$ of $\pi_1^* U(\mathcal{H}_1) \cong \pi_2^* U(\mathcal{H}_2)$. However, the definition depends upon the isomorphism $\pi_1^* U(\mathcal{H}_1) \cong \pi_2^* U(\mathcal{H}_2)$ the choice of which is unique only up to a character $\chi \in \mathrm{Hom}(\Gamma, \mathbb{T})$ and is therefore canonical only when the latter group is trivial. The following examples should illustrate our discussion

(i) The trivial and spin $\frac{1}{2}$ representations V_0, V_1 of SU_2 may be regarded as projective representations of $\mathrm{SO}_3 = \mathrm{SU}_2 / \{\pm 1\}$. However, their direct sum does not factor through SO_3 since $\pi_0(-1) = 1$, $\pi_1(-1) = -1$. The difficulty is that $\pi_0^* U(V_1) \cong \mathrm{SO}_3 \times \mathbb{T} \ncong \mathrm{SU}_2 \times \mathbb{T}/(-1, -1) \cong \pi_1^* U(V_2)$.

(ii) The projectivisations of $z \to \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix}$ give, for varying $n \in \mathbb{Z}$ inequivalent projective representations of $\mathbb{T}$ which are a direct sum of two copies of the unique irreducible projective representation of $\mathbb{T}$.

2. Level 1 representations of LG

We consider in this section the lowest energy subspaces of the level 1 positive energy representations of LG. When $G = \mathrm{Spin}_{2n}$ or is more generally *simply–laced, i.e.* all roots have equal length, these are exactly the *minimal* G–modules. Their simple weight structure and tensor product rules are described in §2.2. They are used in §2.3 to prove that any level ℓ representation of $L\,\mathrm{Spin}_{2n}$ occurs as a summand in an ℓ–fold tensor product $\mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_\ell}$ of level 1 representations.

2.1. Lattices and Lie groups.

We begin by gathering some elementary properties of the lattices canonically associated to G. The present discussion follows [**GO1**]. Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. By the *roots* of G we shall always mean its infinitesimal roots, namely the set R of linear forms $\alpha \in i\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, i\mathbb{R})$ such that the subspace

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}_\mathbb{C} |\, [h,x] = \alpha(h)x\ \forall h \in \mathfrak{t}_\mathbb{C}\} \tag{2.1.1}$$

is non–zero. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of R and θ the corresponding highest root. The basic inner product $\langle\cdot,\cdot\rangle$ is positive definite on $i\mathfrak{t}$ and gives an identification $i\mathfrak{t}^* \cong i\mathfrak{t}$ of which we shall make implicit use. The *coroots* of G are the elements of $i\mathfrak{t}$ given by $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$. They form the dual root system $R^\vee$.

The *root* and *coroot* lattices $\Lambda_R \subset i\mathfrak{t}^*$, $\Lambda_R^\vee \subset i\mathfrak{t}$ are the lattices spanned by R and $R^\vee$ respectively. They have $\mathbb{Z}$-basis given by Δ and $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$. Since θ is a long root and there are at most two root lengths in R with the ratio of the squared length of a long root by that of a short root equal to 2 or 3, rewriting $\alpha^\vee = \frac{\langle\theta,\theta\rangle}{\langle\alpha,\alpha\rangle}\alpha$ we see that $\Lambda_R^\vee \subset \Lambda_R$. Notice that $\langle\alpha^\vee, \alpha^\vee\rangle = \frac{4}{\langle\alpha,\alpha\rangle} = 2\frac{\langle\theta,\theta\rangle}{\langle\alpha,\alpha\rangle}$ so that $\Lambda_R^\vee$ is an even, and therefore integral lattice. The *weight* and *coweight* lattices $\Lambda_W \subset i\mathfrak{t}^*$, $\Lambda_W^\vee \subset i\mathfrak{t}$ are the lattices dual to $\Lambda_R^\vee$ and Λ_R respectively. They have $\mathbb{Z}$-basis given by the *fundamental (co)weights* λ_i, $\lambda_i^\vee$ which are defined by

$$\langle\lambda_i, \alpha_j^\vee\rangle = \langle\lambda_i^\vee, \alpha_j\rangle = \delta_{ij} \tag{2.1.2}$$

Clearly, $\Lambda_W^\vee \subset \Lambda_W$. Moreover, by the integrality properties of root systems, $\langle\alpha, \beta^\vee\rangle \in \mathbb{Z}$ for any root α and coroot $\beta^\vee$ so that $\Lambda_R \subset \Lambda_W$ and, dually, $\Lambda_R^\vee \subset \Lambda_W^\vee$. Graphically,

$$\begin{array}{ccccc} \Lambda_R & \subset & \Lambda_W & \subset & i\mathfrak{t}^* \\ \cup & & \cup & & \\ \Lambda_R^\vee & \subset & \Lambda_W^\vee & \subset & i\mathfrak{t} \end{array} \tag{2.1.3}$$

Let $Z(G)$ be the centre of G and $\widehat{Z(G)} = \mathrm{Hom}(Z(G), \mathbb{T})$ its Pontriagin dual. Then,

LEMMA 2.1.1.

(i) *The map* $e(h) = \exp_T(-2\pi i h)$ *induces an isomorphism* $\Lambda_W^\vee/\Lambda_R^\vee \cong Z(G)$.

(ii) *The pairing* $\mu(\exp_T(h)) = e^{\langle\mu,h\rangle}$ *induces an isomorphism* $\Lambda_W/\Lambda_R \cong \widehat{Z(G)}$.

PROOF. (i) Since G is connected, T is maximal abelian and therefore $Z(G) \subset T$. It follows that $Z(G) \cong e^{-1}(Z(G))/\operatorname{Ker} e$ where the integral lattice $\operatorname{Ker} e \cong \mathrm{Hom}(\mathbb{T}, T)$ is equal to $\Lambda_R^\vee$ since G is simply–connected [**Ad**, thm. 5.47]. To show that $e(h) \in Z(G)$ iff $h \in \Lambda_W^\vee$, we use $Z(G) = \mathrm{Ker}(\mathrm{Ad}_G)$ and the fact that if $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, then $\mathrm{Ad}(e(h))x_\alpha = \exp(-2\pi i\,\mathrm{ad}(h))x_\alpha = e^{-2\pi i\alpha(h)}x_\alpha$ is equal to x_α iff $\alpha(h) \in \mathbb{Z}$.

(ii) The map $\Lambda_W/\Lambda_R \to \widehat{\Lambda_W^\vee/\Lambda_R^\vee}$, $\mu \to e^{-2\pi i\langle\mu,\cdot\rangle}$ is readily seen to be an isomorphism and coincides with the given pairing under the identification $Z(G) \cong \Lambda_W^\vee/\Lambda_R^\vee$ ◇

REMARK. When G is simply–laced, *i.e.* with roots of equal length, the basic inner product identifies roots and coroots and the vertical inclusions in (2.1.3) are equalities. Moreover, lemma 2.1.1 yields a a canonical isomorphism $\widehat{Z(G)} \cong Z(G)$.

The Weyl group W of G is the finite group generated in $\mathrm{End}(i\mathfrak{t}^*)$ by the orthogonal reflections σ_α corresponding to the roots $\alpha \in R$. Since

$$\sigma_\alpha(\mu) = \mu - 2\frac{\langle \mu, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha = \mu - \langle \mu, \alpha^\vee\rangle\alpha = \mu - \langle \mu, \alpha\rangle\alpha^\vee \tag{2.1.4}$$

the action of W preserves $\Lambda_R^\vee$–cosets in $\Lambda_W^\vee$ and Λ_R–cosets in Λ_W. Call $\mu \in \Lambda_W^\vee$ (resp. $\mu \in \Lambda_W$) *minimal* if it is of minimal length in its $\Lambda_R^\vee$ (resp. Λ_R)–coset. The following gives a characterisation of minimal (co)weights.

PROPOSITION 2.1.2. *There is, in each $\Lambda_W^\vee/\Lambda_R^\vee$–coset (resp. Λ_W/Λ_R–coset) a unique W–orbit of elements of minimal length. These may equivalently be characterised as those λ such that*

$$\langle \lambda, \alpha\rangle \in \{0, \pm 1\} \qquad \textit{(resp. } \langle \lambda, \alpha^\vee\rangle \in \{0, \pm 1\}\textit{)} \tag{2.1.5}$$

for any root α (resp. coroot $\alpha^\vee$).

PROOF. It is sufficient to consider the case of $\Lambda_W^\vee/\Lambda_R^\vee$ since Λ_R, Λ_W are the coroot and coweight lattices of the dual root system $R^\vee$. Let $\mu \in \Lambda_W^\vee$ be of minimal length in its $\Lambda_R^\vee$–coset. Then, for any root β and corresponding coroot $\beta^\vee = \frac{2\beta}{\langle \beta, \beta\rangle}$, we have $\|\mu \pm \beta^\vee\|^2 \geq \|\mu\|^2$ and, expanding $|\langle \mu, \beta\rangle| \leq 1$. Assume that $\lambda \in \Lambda_W^\vee$ satisfies (2.1.5) and $\nu = \lambda \mod \Lambda_R^\vee$ is of minimal length in its coset. We claim that $w\lambda = \nu$ for an appropriate $w \in W$. To see this, write $\nu = \lambda + \beta_1^\vee + \cdots + \beta_r^\vee$ where the $\beta_i^\vee$ are (possibly repeated) coroots. Clearly, one cannot have $\langle \lambda, \beta_i\rangle \geq 0$ for all i otherwise

$$\langle \nu, \nu\rangle = \langle \lambda, \lambda\rangle + \langle \sum \beta_i^\vee, \sum \beta_i^\vee\rangle + 2\langle \lambda, \sum \beta_i^\vee\rangle > \langle \lambda, \lambda\rangle \tag{2.1.6}$$

in contradiction with the minimality of ν. Thus, by (2.1.5) there exists an $i \in \{1, \ldots, r\}$ such that $\langle \lambda, \beta_i\rangle = -1$ and therefore $\lambda_1 := \sigma_{\beta_i}\lambda = \lambda + \beta_i^\vee$. Moreover, λ_1 satisfies (2.1.5) since W permutes the roots and preserves $\langle \cdot, \cdot\rangle$. We may therefore iterate the above step to find a permutation τ of $\{1, \ldots, r\}$ such that

$$\lambda_i := \lambda + \beta_{\tau(1)}^\vee + \cdots + \beta_{\tau(i)}^\vee = \sigma_{\beta_{\tau(i)}} \cdots \sigma_{\beta_{\tau(1)}}\lambda \tag{2.1.7}$$

In particular, $\lambda_r = \nu$ and therefore $\nu \in W\lambda$ whence $\|\lambda\| = \|\nu\|$ $\diamond$

Recall that a weight $\mu \in \Lambda_W$ is *dominant* if it lies in the cone

$$\Lambda_W^+ = \{\nu \in \Lambda_W |\, \langle \nu, \alpha_i^\vee\rangle \geq 0\ \forall \alpha_i^\vee \in \Delta^\vee\} = \bigoplus \lambda_i \mathbb{N} \tag{2.1.8}$$

Since Λ_W^+ is a fundamental domain for the action of W on Λ_W, lemma 2.1.1 and proposition 2.1.2 establish a bijective correspondence between elements in $\widehat{Z(G)}$ and minimal dominant weights. Dually, the elements of $Z(G)$ correspond to the minimal dominant coweighs, *i.e.* those $\mu \in \Lambda_W^\vee$ of minimal length in their $\Lambda_R^\vee$–coset lying in $\bigoplus \lambda_i^\vee \cdot \mathbb{N}$.

2.2. Minimal G–modules.

An irreducible G–module is *minimal* if its highest weight μ is a minimal (dominant) weight. The following is proved in [**GO1**]

PROPOSITION 2.2.1. *Let V be an irreducible G–module. Then V is minimal if, and only if its weights lie in a single orbit of the Weyl group and therefore have multiplicity one.*

PROOF. We use the fact that the set of weights $\Pi(V)$ of V is the union of the W–orbits of its highest weight λ and any other dominant weight μ differing from λ by a sum of negative roots [**Hu**, §21.3]. If λ is of minimal length in its Λ_R–coset and $\pi \neq 0$ is a sum of positive roots such that $\lambda - \pi$ is dominant, then $\langle \lambda - \pi, \pi\rangle \geq 0$ and in particular $\langle \lambda, \pi\rangle > 0$. Thus

$$\|\lambda - \pi\|^2 = \|\lambda\|^2 - \langle \lambda - \pi, \pi\rangle - \langle \pi, \lambda\rangle < \|\lambda\|^2 \tag{2.2.1}$$

a contradiction. Conversely, assume that $\Pi(V) = W\lambda$ and let $\mu \in \lambda + \Lambda_R$ be dominant and of minimal length. We claim that $\mu = \lambda$. To see this, write $\mu - \lambda = \sum k_i\alpha_i = \pi - \nu$ where the α_i are simple roots, $\pi = \sum_{i:k_i>0} k_i\alpha_i$ and $\nu = \sum_{i:k_i<0} -k_i\alpha_i$. Notice that $\langle \pi, \nu\rangle \leq 0$ since distinct simple roots form an obtuse angle. It follows that

$$\|\mu\|^2 = \|\lambda - \nu\|^2 + 2\langle \lambda - \nu, \pi\rangle + \|\pi\|^2 \geq \|\lambda - \nu\|^2 + \|\pi\|^2 \tag{2.2.2}$$

whence $\pi = 0$ by minimality of μ. Thus, $\mu = \lambda - \nu$ and therefore $\nu = 0$ since μ is dominant and $\Pi(V) = W\lambda$ $\diamond$

The following is an instance of the Brauer–Weyl rules for computing tensor products [**Hu**, ex. 9, §24.4]

Proposition 2.2.2. *Let V_μ, V_λ be the irreducible G–modules with highest weights μ, λ. If λ is minimal, then*

$$V_\mu \otimes V_\lambda = \bigoplus_\nu V_{\mu+\nu} \tag{2.2.3}$$

where ν ranges over those weights of V_λ such that $\mu + \nu$ is dominant.

Proof. Let $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ be the half–sum of the positive roots so that $\langle \rho, \alpha_i^\vee\rangle = 1$ for any simple coroot $\alpha_i^\vee$. By the Weyl character formula, the character of V_μ is

$$\chi_\mu = \frac{A(\mu + \rho)}{\delta} \tag{2.2.4}$$

where $A(\beta) = \sum_{w\in W}(-1)^w e^{w\beta}$ with $e^\beta(\exp_T(h)) = e^{\beta(h)}$ and $\delta = \prod_{\alpha>0}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})$. On the other hand, by proposition 2.2.1

$$\chi_\lambda = \frac{1}{N_\lambda}\sum_{w\in W} e(w\lambda) \qquad \text{where} \qquad N_\lambda = |\{w \in W | w\lambda = \lambda\}| \tag{2.2.5}$$

It follows that

$$\chi_\mu\chi_\lambda = \frac{1}{\delta N_\lambda}\sum_{w'\in W}(-1)^{w'} e(w'(\mu + \rho))\sum_{w\in W} e(w'w\lambda) = \frac{1}{N_\lambda}\sum_{w\in W}\frac{A(\mu + w\lambda + \rho)}{\delta} \tag{2.2.6}$$

If $\mu + w\lambda$ is not dominant then, for some simple coroot $\alpha_i^\vee$,

$$0 > \langle \mu + w\lambda, \alpha_i^\vee\rangle = \langle \mu, \alpha_i^\vee\rangle + \langle w\lambda, \alpha_i^\vee\rangle \tag{2.2.7}$$

so that $\langle \mu, \alpha_i^\vee\rangle = 0$ and $\langle w\lambda, \alpha_i^\vee\rangle = -1$ since μ is dominant and $w\lambda$ minimal and therefore satisfies (2.1.5). Thus, $\langle \mu + w\lambda + \rho, \alpha_i^\vee\rangle = 0$ and if $\sigma_i \in W$ is the simple reflection corresponding to $\alpha_i^\vee$, then

$$A(\mu + w\lambda + \rho) = -A(\sigma_i(\mu + w\lambda + \rho)) = -A(\mu + w\lambda + \rho) = 0 \tag{2.2.8}$$

whence, denoting the set of weight of V_λ by $\Pi(V_\lambda)$

$$\chi_\mu\chi_\lambda = \frac{1}{N_\lambda}\sum_{w\in W:\mu+w\lambda\in\Lambda_W^+}\chi_{\mu+w\lambda} = \sum_{\nu\in\Pi(V_\lambda):\mu+\nu\in\Lambda_W^+}\chi_{\mu+\nu} \tag{2.2.9}$$

$\diamond$

Corollary 2.2.3. *Let V_i, V_k, V_j be irreducible G–modules one of which is minimal. Then,* $\mathrm{Hom}_G(V_i \otimes V_k, V_j)$ *is at most one–dimensional.*

Proof. This follows from proposition 2.2.2 if V_i or V_k are minimal. If V_j is minimal, so is V_j^* by proposition 2.2.1 and therefore $\mathrm{Hom}_G(V_i \otimes V_k, V_j) \cong \mathrm{Hom}_G(V_i \otimes V_j^*, V_k^*)$ is at most one–dimensional $\diamond$

2.3. Level 1 representations of $L\,\mathrm{Spin}_{2n}$.

Consider now $G = \mathrm{Spin}_{2n}$, $n \geq 3$. Let $\mathcal{H}$ be an irreducible, level 1 positive energy representation of LG. Its lowest energy subspace $\mathcal{H}(0)$ is an irreducible Spin_{2n}–module whose highest weight λ satisfies $\langle \lambda, \theta \rangle \leq 1$. Denote by θ_i, $i = 1 \ldots n$ an orthonormal basis of $\mathbb{R}^n$ and identify the simple roots of Spin_{2n} with the vectors $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1 \ldots n-1$ and $\alpha_n = \theta_{n-1} + \theta_n$. The corresponding highest root is $\theta = \theta_1 + \theta_2$ and, by inspection [2], λ is one of $\{0, \lambda_1, \lambda_{n-1}, \lambda_n\}$ where λ_i is the fundamental weight corresponding to α_i so that

$$\lambda_1 = v = \theta_1 \qquad \lambda_{n-1} = s_+ = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} - \theta_n) \qquad \lambda_n = s_- = \frac{1}{2}(\theta_1 + \cdots + \theta_n) \tag{2.3.1}$$

The corresponding irreducible representations are the trivial, vector and spin representations. Notice that the above weights are minimal. Indeed, since Spin_{2n} is simply–laced, $\theta^\vee = \theta$ is the highest coroot and therefore, for any coroot $\alpha^\vee$

$$\langle \lambda, \alpha^\vee \rangle = \langle \lambda, \theta \rangle - \langle \lambda, \theta - \alpha^\vee \rangle \leq 1 \tag{2.3.2}$$

since $\theta - \alpha^\vee$ is a sum of positive coroots. By proposition 2.1.2, λ is a minimal dominant weight. Since the Weyl group acts by permutation and even numbers of sign changes of the θ_i, it follows by proposition 2.2.1 that the weights of the corresponding irreducible representations are

$$\Pi(V_0) = \{0\} \tag{2.3.3}$$

$$\Pi(V_v) = \{\pm\theta_i\} \tag{2.3.4}$$

$$\Pi(V_{s_+}) = \{\frac{1}{2}\sum \epsilon_i\theta_i |\, \epsilon_i \in \{\pm 1\} \text{ and } \epsilon_i = -1 \text{ for an odd number of } i\} \tag{2.3.5}$$

$$\Pi(V_{s_-}) = \{\frac{1}{2}\sum \epsilon_i\theta_i |\, \epsilon_i \in \{\pm 1\} \text{ and } \epsilon_i = -1 \text{ for an even number of } i\} \tag{2.3.6}$$

and that their tensor product rules are governed by proposition 2.2.2.

LEMMA 2.3.1. *An irreducible* Spin_{2n}*–module V is admissible at level ℓ if, and only if it arises as a summand of an ℓ–fold tensor product $V_{i_1} \otimes \cdots \otimes V_{i_\ell}$ whose factors are admissible at level 1.*

PROOF. Let λ be the highest weight of V and denote by V_μ the irreducible G–module with highest weight μ. If V is contained in $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$ where $\langle \lambda_i, \theta \rangle \leq 1$, then $\lambda = \sum_i \lambda_i - \pi$ where π is a sum of positive roots and therefore $\langle \lambda, \theta \rangle \leq \ell$. Conversely, assume that $\langle \lambda, \theta \rangle \leq \ell$. If the inequality is strict, then by induction, $V \subset V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{\ell-1}} \otimes \mathbb{C}$ with $\langle \lambda_i, \theta \rangle \leq 1$. Suppose therefore that $\langle \lambda, \theta \rangle = \ell \geq 2$ and write $\lambda = \sum_i \lambda_i \theta_i$. If $\lambda_2 = 0$, then $\mu = \lambda - \theta_1$ is dominant and satisfies $\langle \mu, \theta \rangle \leq \ell - 1$. Moreover, $V \subset V_\mu \otimes V_{\theta_1}$. If, on the other hand, $\lambda_2 > 0$ and the coordinates of λ are of the form $\lambda_1 = \lambda_2 = \cdots = \lambda_m > \lambda_{m+1} \geq \cdots \geq \lambda_n$ then $\mu = \lambda - \frac{1}{2}(\theta_1 + \cdots + \theta_m - \theta_{m+1} - \cdots - \theta_n)$ is dominant and satisfies $\langle \mu, \theta \rangle \leq \ell - 1$. Since $\frac{1}{2}(\theta_1 + \cdots + \theta_m - \theta_{m+1} - \cdots - \theta_n)$ is a weight of one of the two spin representations, call it V_σ, proposition 2.2.2 yields $V \subset V_\mu \otimes V_\sigma$. The result now follows by induction $\diamond$

LEMMA 2.3.2. *Let $(\pi, \mathcal{H})$ be a positive energy representation of LG and $V \subset \mathcal{H}(0)$ an irreducible G-module. Then, the closure of the span of $\pi(LG)V$ is unitarily equivalent to $\mathcal{H}_V$.*

PROOF. Let $\mathcal{K} \subset \mathcal{H}^{\text{fin}}$ be the subspace generated by V under the action of $L^{\text{pol}}\mathfrak{g}_{\mathbb{C}}$. Its lowest energy subspace is V since any element in the enveloping algebra of $L^{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ may be written as a sum of monomials of the form $X^- X^0 X^+$ where the $X^\mp$ are a product of $x(n)$, with $n \lessgtr 0$ and $X^0 \in \mathfrak{U}\mathfrak{g}$, and V is annihilated by X^+ and left invariant by X^0. It follows that $\mathcal{K}$ is an irreducible $L^{\text{pol}}\mathfrak{g} \rtimes \mathbb{C}d$–module. Indeed, any submodule $\mathcal{V} \subset \mathcal{K}$ is necessarily graded and the corresponding orthogonal projection P commutes with $\mathrm{Rot}\, S^1$. It therefore maps $V = \mathcal{K}(0)$ into V and since P commutes with $\mathfrak{g}$ we have $PV = 0$ or V. Thus, $\mathcal{V} = P\mathfrak{U}L^{\text{pol}}\mathfrak{g}_{\mathbb{C}}V = \mathfrak{U}L^{\text{pol}}\mathfrak{g}_{\mathbb{C}}PV$ is 0 or $\mathcal{K}$. By proposition 1.2.2, $\overline{\mathcal{K}}$ is invariant and irreducible under LG since its finite energy subspace is $\mathcal{K}$ and it follows that it is unitarily equivalent to $\mathcal{H}_V$. Since $\overline{\mathcal{K}}$ contains the closure of the span of $\pi(LG)V$ it coincides with it $\diamond$

The following useful result is due to Pressley and Segal [**PS**, prop. 9.3.9]

[2] see also the tables in §3.3 or [**Bou**].

PROPOSITION 2.3.3. *Any irreducible, level ℓ positive energy representation $\mathcal{H}$ of $L\,\mathrm{Spin}_{2n}$ is a summand in an ℓ–fold tensor product $\mathcal{H}_{i_1}\otimes\cdots\otimes\mathcal{H}_{i_\ell}$ of level 1 representations.*

PROOF. By lemma 2.3.1, the lowest energy subspace $\mathcal{H}(0)$ of $\mathcal{H}$ is contained in some tensor product $V_{i_1}\otimes\cdots\otimes V_{i_\ell}$ of minimal representations of Spin_{2n}. Let $\mathcal{H}_{i_k}$ be the irreducible level 1 representations whose lowest energy subspace are the V_{i_k}. The lowest energy subspace of the level ℓ representation $\mathcal{H}_{i_1}\otimes\cdots\otimes\mathcal{H}_{i_\ell}$ contains $\mathcal{H}(0)$ and therefore, by lemma 2.3.2, the closure of the linear span of $L\,\mathrm{Spin}_{2n}\,\mathcal{H}(0)$ inside $\mathcal{H}_{i_1}\otimes\cdots\otimes\mathcal{H}_{i_\ell}$ is isomorphic to $\mathcal{H}$ $\diamond$

REMARK. Lemma 2.3.1 holds for any simply–connected classical Lie group G. This may be established for $G=\mathrm{Spin}_{2n+1}$ by an almost identical proof to that of lemma 2.3.1 and for SU_n and Sp_n by simply noticing that all fundamental weights satisfy $\langle\lambda,\theta\rangle\leq 1$. Thus, proposition 2.3.3 holds for the loop groups of all classical Lie groups. Notice that lemma 2.3.1 does not hold for $G=E_8$ since the only irreducible module admissible at level 1 is the trivial representation.

3. Discontinuous loops and outer automorphisms of LG

We consider in this section the automorphic action on LG of the group of *discontinuous loops*

$$L_ZG=\{f\in C^\infty(\mathbb{R},G)|\,f(x+2\pi)f(x)^{-1}\in Z(G)\} \tag{3.1}$$

We show in §3.2 that the category $\mathcal{P}_\ell$ of positive energy representations of LG at level ℓ is closed under conjugation by L_ZG. The corresponding abstract action of $Z(G)=L_ZG/LG$ on the level ℓ alcove of G which parametrises the irreducibles in $\mathcal{P}_\ell$ coincides with the geometric one obtained by realising $Z(G)$ as a distinguished subgroup of the automorphisms of the extended Dynkin diagram of G. We begin by studying the latter. An explicit description of this action according to the Lie type of G may be found in §3.3.

3.1. Geometric action of $Z(G)$ on the level ℓ alcove.

This subsection is essentially an expanded version of [**Bou**, ch. VI, §2.3]. The notation follows that of the section 2.

LEMMA 3.1.1. *There is a bijective correspondence between elements of $Z(G)\backslash\{1\}$ and fundamental coweights corresponding to special roots, i.e. the $\alpha_i\in\Delta$ bearing the coefficient 1 in the expansion*

$$\theta=\sum m_i\alpha_i \tag{3.1.1}$$

PROOF. By proposition 2.1.2, $\mu\in(\Lambda_W^\vee)^+$ is minimal iff $\langle\mu,\theta\rangle\leq 1$. Indeed, for any positive root α, we get $0\leq\langle\mu,\alpha\rangle\leq\langle\mu,\theta\rangle-\langle\mu,\theta-\alpha\rangle\leq\langle\mu,\theta\rangle$. Since $\langle\mu,\theta\rangle=0$ implies $\mu=0$, the non–zero minimal dominant coweights are those $\mu\in(\Lambda_W^\vee)^+$ such that $\langle\mu,\theta\rangle=1$. Writing $\mu=\sum_i k_i\lambda_i^\vee$, $k_i\geq 0$ and using (3.1.1), we find $\langle\mu,\theta\rangle=\sum k_im_i$. Since $\theta-\alpha_i$ is a sum of positive roots, $m_i\geq 1$ for any i and result follows $\diamond$

Denote $-\theta$ by α_0, then

LEMMA 3.1.2. *For any special root α_i, the set $\Delta_i=\Delta\backslash\{\alpha_i\}\cup\{\alpha_0\}$ is a basis of R with highest root $-\alpha_i$ and dual basis*

$$\lambda_0^{\vee\prime}=-\lambda_i^\vee \tag{3.1.2}$$

$$\lambda_j^{\vee\prime}=\lambda_j^\vee-\langle\theta,\lambda_j^\vee\rangle\lambda_i^\vee \tag{3.1.3}$$

PROOF. Let $x\in\mathfrak{t}_\mathbb{C}$, then

$$x=\sum\langle x,\lambda_j^\vee\rangle\alpha_j=\langle x,\lambda_i^\vee\rangle\theta+\sum_{j\neq i}(\langle x,\lambda_j^\vee\rangle-\langle x,\lambda_i^\vee\rangle\langle\theta,\lambda_j^\vee\rangle)\alpha_j \tag{3.1.4}$$

so that Δ_i is a vector space basis of $\mathfrak{t}_\mathbb{C}$ with dual basis given by (3.1.2)–(3.1.3). If $0<\beta\in R$ then either $\langle\beta,\lambda_i^\vee\rangle=0$ in which case $\langle\beta,\lambda_0^{\vee\prime}\rangle$ and $\langle\beta,\lambda_j^{\vee\prime}\rangle$ are all non-negative or $\langle\beta,\lambda_i^\vee\rangle=1$ since $\lambda_i^\vee$ is a minimal dominant coweight. In the latter case $\langle\beta,\lambda_0^{\vee\prime}\rangle=-1$ and $\langle\beta,\lambda_j^{\vee\prime}\rangle=\langle\beta-\theta,\lambda_j^\vee\rangle\leq 0$. Thus,

Δ_i is a basis of R. Next, for any $\beta \in R$, (2.1.5) yields $\langle -\alpha_i - \beta, \lambda_0^{\vee\prime} \rangle = 1 + \langle \beta, \lambda_i^\vee \rangle \geq 0$ since $\lambda_i^\vee$ is minimal. Moreover, for $j \neq i$

$$\langle -\alpha_i - \beta, \lambda_j^{\vee\prime} \rangle = \langle \theta, \lambda_j^\vee \rangle - \langle \beta, \lambda_j^\vee \rangle + \langle \theta, \lambda_j^\vee \rangle \langle \beta, \lambda_i^\vee \rangle = \langle \theta - \beta, \lambda_j^\vee \rangle + \langle \theta, \lambda_j^\vee \rangle \langle \beta, \lambda_i^\vee \rangle \tag{3.1.5}$$

The above is clearly non-negative if $\langle \beta, \lambda_i^\vee \rangle \geq 0$. If, on the other hand $\langle \beta, \lambda_i^\vee \rangle = -1$ so that $\beta < 0$ then (3.1.5) is equal to $-\langle \beta, \lambda_j^\vee \rangle \geq 0$ and $-\alpha_i$ is the highest root relative to Δ_i ◇

Proposition 3.1.3. *Let $\overline{\Delta} = \Delta \cup \{\alpha_0\}$. Then, for any special root α_i, there exists a unique*

$$w_i \in W_0 = \{w \in W |\, w\overline{\Delta} = \overline{\Delta}\} \tag{3.1.6}$$

such that $w\alpha_0 = \alpha_i$. The resulting map $\imath : Z(G) \rightarrow W_0$ obtained by identifying $Z(G) \setminus \{1\}$ with the set of special roots is an isomorphism.

Proof. The existence of w_i follows from the previous lemma since W acts transitively on the set of basis of R and maps highest roots to highest roots. w_i is unique because an element $w \in W_0$ is determined by $w\alpha_0$. Indeed, if $w_1\alpha_0 = \alpha_j = w_2\alpha_0$, then $w_2^{-1}w_1$ is a permutation of Δ and is therefore the identity since W acts simply on basis. $\imath$ is injective because $w_i\alpha_0 = \alpha_i$. Let now $w \in W_0$. We claim that $\alpha_i = w\alpha_0$ is a special root. It then follows by uniqueness that $w = w_i$ and therefore that $\imath$ is surjective. To see this, we apply w to (3.1.1) and get $-\alpha_i = \sum_j m_j w\alpha_j$ while at the same time $-\alpha_i = m_i^{-1}(\alpha_0 + \sum_{j\neq i} m_j\alpha_j)$. Comparing the coefficients of α_0 we get $m_i = 1$. To prove that $\imath$ is a homomorphism, let α_i and α_j be special roots. Then either $w_iw_j = 1$ or $w_iw_j = w_k$ where α_k is another special root. In the former case, $w_i\alpha_j = \alpha_0$ and therefore, by (3.1.2)

$$w_i\lambda_j^\vee = \lambda_0^{\vee\prime} = -\lambda_i^\vee \tag{3.1.7}$$

so that $\lambda_j^\vee = -\lambda_i^\vee \mod \Lambda_R^\vee$ since W leaves $\Lambda_W^\vee/\Lambda_R^\vee$ cosets invariant. In the latter, $w_i\alpha_j = \alpha_k$ and therefore, using (3.1.3)

$$w_i\lambda_j^\vee = \lambda_k^{\vee\prime} = \lambda_k^\vee - \langle \theta, \lambda_k^\vee \rangle \lambda_i^\vee = \lambda_k^\vee - \lambda_i^\vee \tag{3.1.8}$$

whence $\lambda_i^\vee + \lambda_j^\vee = \lambda_k^\vee \mod \Lambda_R^\vee$ ◇

The following is well–known and often rediscovered [**OT**]

Corollary 3.1.4. *$Z(G)$ is canonically isomorphic to the group of automorphisms of the extended Dynkin diagram of G induced by Weyl group elements.*

Proposition 3.1.5. *For any $\ell \in \mathbb{N}$, there is a canonical action of $Z(G)$ on the level ℓ alcove $\mathcal{A}_\ell$ given by*

$$z \longrightarrow A_i = \tau(\ell\lambda_i^\vee)w_i \tag{3.1.9}$$

where i is the index of the special root corresponding to z via lemma 3.1.1, τ denotes translation and $w_i = \imath(z)$ corresponds to z via proposition 3.1.3.

Proof. The level ℓ alcove is given by

$$\mathcal{A}_\ell = \{\lambda \in \Lambda_W | \langle \lambda, \alpha_i \rangle \geq 0,\ \langle \lambda, \theta \rangle \leq \ell\} \tag{3.1.10}$$

If $\lambda \in \mathcal{A}_\ell$ then for $j \neq i$, $\langle A_i\lambda, \alpha_j \rangle = \langle \lambda, w_i^{-1}\alpha_j \rangle \geq 0$ since $w_i^{-1}\alpha_j \neq \alpha_0$. On the other hand, $\langle A_i\lambda, \alpha_i \rangle = \ell + \langle \lambda, \alpha_0 \rangle \geq 0$. Finally, $\langle A_i\lambda, \theta \rangle = \ell - \langle \lambda, w_i^{-1}\theta \rangle \leq \ell$ so that the A_i leave $\mathcal{A}_\ell$ invariant. Next, $A_iA_j = \tau(\ell(\lambda_i^\vee + w_i\lambda_j^\vee))w_iw_j$. If $w_iw_j = 1$, we get by (3.1.7) and the previous proposition $A_iA_j = 1$. If on the other hand $w_iw_j = w_k$, (3.1.8) yields $A_iA_j = A_k$ ◇

Corollary 3.1.6. *If G is simply–laced, the canonical action of $Z(G)$ on the level 1 alcove*

$$\mathcal{A}_1 \cong \widehat{Z(G)} \cong Z(G) \tag{3.1.11}$$

coincides with left multiplication. In particular, it is transitive and free.

PROOF. Let $\lambda \in \mathcal{A}_1$. Since G is simply–laced, $\theta^\vee = \theta$ is the highest coroot and therefore, for any coroot $\alpha^\vee$,

$$\langle \lambda, \alpha^\vee \rangle = \langle \lambda, \theta \rangle - \langle \lambda, \theta - \alpha^\vee \rangle \leq \langle \lambda, \theta \rangle \leq 1 \tag{3.1.12}$$

since $\theta - \alpha^\vee$ is a sum of positive coroots. Thus, by proposition 2.1.2 the points in $\mathcal{A}_1$ are the minimal dominant weights and therefore are in one–to–one correspondence with elements in $\widehat{Z(G)}$. As previously remarked, the basic inner product identifies $\Lambda_R^\vee, \Lambda_W^\vee$ with Λ_R, Λ_W respectively and therefore determines a natural isomorphism $Z(G) \cong \widehat{Z(G)}$. Let now $z \in Z(G) \setminus \{1\}$ correspond to the fundamental coweight $\lambda_i^\vee$. Then, if $\lambda \in \mathcal{A}_1$, we have by (3.1.9),

$$A_i \lambda = \lambda_i^\vee + w_i \lambda = \lambda_i^\vee + \lambda + (w_i \lambda - \lambda) \tag{3.1.13}$$

Since W preserves Λ_R–cosets in Λ_W, $A_i \lambda = \lambda_i^\vee + \lambda \mod \Lambda_R$ as claimed $\diamond$

3.2. Action of $Z(G)$ on the positive energy representations.

The group of discontinuous loops $L_Z G$ defined by (3.1) acts automorphically on LG by conjugation. This induces an action on the set of equivalence classes of projective unitary representations of LG given by $\zeta_* \pi(\gamma) = \pi(\zeta^{-1}\gamma\zeta)$ for any $\zeta \in L_Z G$. Clearly, $(\zeta_1\zeta_2)_*\pi = \zeta_{1*}\zeta_{2*}\pi$ and this action factors through $Z = L_Z G/LG$ since $\gamma_*\pi(\cdot) = \pi(\gamma)^*\pi(\cdot)\pi(\gamma)$ whenever $\gamma \in LG$. Let $(\pi, \mathcal{H})$ be irreducible and of positive energy and denote by U_θ the corresponding action of $\operatorname{Rot} S^1$. For a fixed $\zeta \in L_Z G$, any intertwining action of $\operatorname{Rot} S^1$ for $\zeta_*\pi$, whether of positive energy or not is necessarily given by

$$V_\theta = \pi(\zeta^{-1}\zeta_\theta) U_\theta \tag{3.2.1}$$

Indeed, $\zeta^{-1}\zeta_\theta \in LG$ and V_θ yields a projective action of $\operatorname{Rot} S^1$ satisfying $V_\theta \zeta_*\pi(\gamma) V_\theta^* = \zeta_*\pi(\gamma_\theta)$. Moreover, if V_θ^i, $i = 1, 2$ are two intertwining actions of $\operatorname{Rot} S^1$, then $W_\theta = (V_\theta^1)^* V_\theta^2$ commutes projectively with the action of LG so that the following holds in $U(\mathcal{H})$ for any $\theta :$ $W_\theta \pi(\gamma) W_\theta^* \pi(\gamma)^* = \chi(\gamma, \theta)$, where $\chi(\gamma, \theta) \in \mathbb{T}$ and depends multiplicatively on $\gamma \in LG$. Since LG is equal to its commutator subgroup [**PS**, propn. 3.4.1.], $\chi \equiv 1$ and it follows by Shur's lemma that $W_\theta = 1$ in $PU(\mathcal{H})$. Thus, $\zeta_*\pi$ is of positive energy iff V_θ is a positive energy representation of $\operatorname{Rot} S^1$.

PROPOSITION 3.2.1. *If $(\pi, \mathcal{H})$ is a positive energy representation of LG and $\zeta \in L_Z G$, the conjugated representation $\zeta_*\pi$ is of positive energy.*

PROOF. It is sufficient to prove the above for a given set of representatives of LG–cosets in $L_Z G$. A particularly convenient choice is obtained via lemma 2.1.1 by considering the discontinuous loops $\zeta_\mu(\theta) = \exp_T(-i\theta\mu)$ where $\mu \in \Lambda_W^\vee \subset i\mathfrak{t}$ is a coweight. If $\mu \in \Lambda_R^\vee$, then $\zeta_\mu \in LG$ and the corresponding action of $\operatorname{Rot} S^1$ may be rewritten, by (3.2.1) as $\pi(\zeta_\mu^{-1}\zeta_{\mu\,\theta}) U_\theta = \pi(\zeta_\mu)^* U_\theta \pi(\zeta_\mu)$ which is clearly of positive energy. The general case $\mu \in \Lambda_W^\vee$ is settled by the following simple observation. Notice first that $\zeta_\mu^{-1}\zeta_{\mu\,\theta} = \zeta_\mu(-\theta) = \exp_T(i\theta\mu)$ since ζ_μ is a homomorphism and write μ as a convex combination of elements in the coroot lattice, $\mu = \sum_{i=1}^m t_i \alpha_i$, $t_i \in (0, 1]$, $\sum_i t_i = 1$, $\alpha_i \in \Lambda_R^\vee$. Recall that on $\mathcal{H}$ we have a unitary lift $\widetilde{\pi}$ of π over $T \times \operatorname{Rot} S^1$. Therefore,

$$\widetilde{\pi}(\exp_T(i\theta\mu))\widetilde{\pi}(\exp_{\operatorname{Rot} S^1}(i\theta d)) = \prod_j \widetilde{\pi}(\exp_T(i\theta t_j \alpha_j))\widetilde{\pi}(\exp_{\operatorname{Rot} S^1}(i\theta t_j d)) \tag{3.2.2}$$

is a lift of $\pi(\zeta_\mu^{-1}\zeta_{\mu\,\theta}) U_\theta$ and the product of m commuting representations of $\mathbb{R}$ which by our previous argument are of positive energy. It follows that $\pi(\zeta_\mu^{-1}\zeta_{\mu\,\theta}) U_\theta$ is of positive energy $\diamond$

PROPOSITION 3.2.2. *Let $(\pi, \mathcal{H})$ be a positive energy representation of LG of level ℓ and $\zeta \in L_Z G$. Then,*

(i) *$\zeta_*\pi$ is of level ℓ.*
(ii) *If $\zeta_\mu(\phi) = \exp_T(-i\phi\mu)$ is the discontinuous loop corresponding to $\mu \in \Lambda_W^\vee$, the subspaces of finite energy vectors of π and $\zeta_*\pi$ coincide.*
(iii) *If $\operatorname{Ad}(\zeta) L^{\mathrm{pol}}\mathfrak{g} = L^{\mathrm{pol}}\mathfrak{g}$ and the finite energy subspaces of π and $\zeta_*\pi$ coincide, the conjugated action of $L^{\mathrm{pol}}\mathfrak{g}$ on $\mathcal{H}^{\mathrm{fin}}$ is given by*

$$\zeta_*\pi(X) = \pi(\zeta^{-1} X \zeta) + i\ell \int_0^{2\pi} \langle \dot{\zeta}\zeta^{-1}, X \rangle \frac{d\theta}{2\pi} \tag{3.2.3}$$

PROOF. It is sufficient to check (i) for the discontinuous loops ζ_μ, $\mu\in\Lambda_W^\vee$. This will be done in the course of the proof of (iii).
(ii) It was remarked in the proof of the previous proposition that the conjugated action of rotations (3.2.1) corresponding to ζ_μ is given by

$$\pi(\zeta_\mu^{-1}\zeta_{\mu_\theta})U_\theta=\pi(\exp_T(i\theta\mu))\pi(\exp_{\mathrm{Rot}\,S^1}(i\theta d)) \tag{3.2.4}$$

which commutes with the original action of $\mathrm{Rot}\,S^1$ given by $U_\theta=\pi(\exp_{\mathrm{Rot}\,S^1}(i\theta d))$. Since both are of positive energy, their finite energy subspaces coincide.
(iii) Let h be the level of $\zeta_*\pi$ and denote by π and $\zeta_*\pi$ the projective representations of $L^{\mathrm{pol}}\mathfrak{g}$ on $\mathcal{H}^{\mathrm{fin}}$ given by theorem 1.2.1 so that

$$[\pi(X),\pi(Y)]=\pi([X,Y])+i\ell B(X,Y) \tag{3.2.5}$$

$$[\zeta_*\pi(X),\zeta_*\pi(Y)]=\zeta_*\pi([X,Y])+ihB(X,Y) \tag{3.2.6}$$

where $B(X,Y)=\int_0^{2\pi}\langle X,\dot{Y}\rangle\frac{d\theta}{2\pi}$ is the fundamental 2-cocycle on $L\mathfrak{g}$. Evidently, $\zeta_*\pi(X)=\pi(\zeta^{-1}X\zeta)+iF(X)$ for some $F(X)\in\mathbb{R}$ since $\zeta_*\pi(\exp_{LG}(X))=\pi(\exp_{LG}(\zeta^{-1}X\zeta))=e^{\pi(\zeta^{-1}X\zeta)}$ in $PU(\mathcal{H})$. It follows that

$$\begin{aligned}
[\zeta_*\pi(X),\zeta_*\pi(Y)]&=[\pi(\zeta^{-1}X\zeta),\pi(\zeta^{-1}Y\zeta)]\\
&=\pi(\zeta^{-1}[X,Y]\zeta)+i\ell\int_0^{2\pi}\langle\zeta^{-1}X\zeta,\zeta^{-1}\dot{Y}\zeta\rangle\frac{d\theta}{2\pi}+i\ell\int_0^{2\pi}\langle\zeta^{-1}X\zeta,(\dot{\zeta^{-1}})Y\zeta+\zeta^{-1}Y\dot{\zeta}\rangle\frac{d\theta}{2\pi}\\
&=\zeta_*\pi([X,Y])-iF([X,Y])+i\ell B(X,Y)+i\ell\int_0^{2\pi}\langle\zeta^{-1}X\zeta,\zeta^{-1}[Y,\dot{\zeta}\zeta^{-1}]\zeta\rangle\frac{d\theta}{2\pi}\\
&=\zeta_*\pi([X,Y])-idF(X,Y)+i\ell B(X,Y)+i\ell\int_0^{2\pi}\langle\dot{\zeta}\zeta^{-1},[X,Y]\rangle\frac{d\theta}{2\pi}
\end{aligned} \tag{3.2.7}$$

where we used the Ad_G invariance of $\langle\cdot,\cdot\rangle$ and the fact that $(\dot{\zeta^{-1}})\zeta+\zeta^{-1}\dot{\zeta}=(\dot{\zeta^{-1}\zeta})=0$. Since hB and ℓB define the same cohomology class iff $h=\ell$, we find by equating the above with (3.2.6) that the level of $\zeta_*\pi$ is ℓ. Moreover, (3.2.3) holds since $[L^{\mathrm{pol}}\mathfrak{g},L^{\mathrm{pol}}\mathfrak{g}]=L^{\mathrm{pol}}\mathfrak{g}$ ◇

THEOREM 3.2.3. *Let $(\pi,\mathcal{H})$ be an irreducible positive energy representation of LG of level ℓ and highest weight λ and $\zeta\in L_ZG$. Then, the conjugated representation $\zeta_*\pi(\gamma)=\pi(\zeta^{-1}\gamma\zeta)$ on $\mathcal{H}$ is of positive energy, level ℓ and highest weight $\zeta\lambda$ where the notation refers to the geometric action of $Z(G)=L_ZG/LG$ on the level ℓ alcove defined by proposition 3.1.5.*

PROOF. As customary, it is sufficient to prove the result for a given choice of representatives of LG–cosets in L_ZG. Let $z\in Z(G)\setminus\{1\}$ correspond to the special root α_j by lemma 3.1.1 and consider the discontinuous loop $\zeta=\zeta_{\lambda_j^\vee}w_j$ where $\lambda_j^\vee$ is the associated fundamental coweight and $w_j\in G$ a representative of the Weyl group element corresponding to z by proposition 3.1.3. Since the action of G commutes with $\mathrm{Rot}\,S^1$ on $\mathcal{H}$, the subspace of finite energy vectors of $\zeta_*\pi$ coincides with that of ${\zeta_{\lambda_j^\vee}}_*\pi$ and, by the previous proposition, with that of π. We may therefore compare the infinitesimal actions of $L^{\mathrm{pol}}\mathfrak{g}_\mathbb{C}$ corresponding to π and $\zeta_*\pi$ on $\mathcal{H}^{\mathrm{fin}}$.

If α is a root and $x_\alpha\in\mathfrak{g}_\alpha$, we have $[\lambda_j^\vee,x_\alpha]=\langle\lambda_j^\vee,\alpha\rangle x_\alpha$. Since $\zeta_{\lambda_j^\vee}(\theta)=\exp_T(-i\lambda_j^\vee\theta)$, this gives

$$\zeta_{\lambda_j^\vee}^{-1}x_\alpha(n)\zeta_{\lambda_j^\vee}(\theta)=x_\alpha\otimes e^{i\theta(n+\langle\lambda_j^\vee,\alpha\rangle)}=x_\alpha(n+\langle\lambda_j^\vee,\alpha\rangle)(\theta) \tag{3.2.8}$$

Thus, $\zeta x_\alpha(n)\zeta^{-1}$ lies in the root space $\mathfrak{g}_{w_j^{-1}\alpha,n+\langle\lambda_j^\vee,\alpha\rangle}$ and therefore, up to a non–zero multiplicative constant

$$\zeta_*\pi(e_\alpha(n))=\pi(e_{w_j^{-1}\alpha}(n+\langle\lambda_j^\vee,\alpha\rangle)) \tag{3.2.9}$$

since no additional term arise from (3.2.3) because $\dot{\zeta}\zeta^{-1}=-i\lambda_j^\vee$ lies in $\mathfrak{t}_\mathbb{C}$ which is orthogonal to $\mathfrak{g}_\alpha$. If, on the other hand $h\in\mathfrak{t}_\mathbb{C}$, then $\zeta^{-1}h(n)\zeta=w_j^{-1}h(n)$ and (3.2.3) reads

$$\zeta_*\pi(h(n))=\pi(w_j^{-1}h(n))+\ell\delta_{n,0}\langle h,\lambda_j^\vee\rangle \tag{3.2.10}$$

Let $\Omega \in \mathcal{H}^{\text{fin}}$ be the highest weight vector for $\zeta_*\pi$. We claim that, up to a scalar factor, $\Omega = \Upsilon$, the highest weight vector for π. To see this, recall that Ω is the unique element of $\mathcal{H}^{\text{fin}}$ annihilated by the subalgebra spanned by the $x(n)$, $x \in \mathfrak{g}_\mathbb{C}$ and $n > 0$ and the $x_\alpha(0)$ with $\alpha > 0$. This in turn is generated by the elements corresponding the simple affine roots, namely $e_{\alpha_i}(0)$ and $e_{\alpha_0}(1)$ where $\alpha_0 = -\theta$. Recalling from proposition 3.1.3 that w_j^{-1} acts as a permutation of $\overline{\Delta} = \{\alpha_0, \ldots, \alpha_n\}$ and maps α_j to α_0, we get, using (3.2.9)

$$\zeta_*\pi(e_{\alpha_k}(0)) = \begin{cases} \pi(e_{w_j^{-1}\alpha_k}(0)) & \text{if } k \neq j \\ \pi(e_{\alpha_0}(1)) & \text{if } k = j \end{cases} \tag{3.2.11}$$

$$\zeta_*\pi(e_{\alpha_0}(1)) = \pi(e_{w_j^{-1}\alpha_0}(0)) \tag{3.2.12}$$

whence $\Omega = \Upsilon$. To find the weight of Ω and therefore the highest weight of $\zeta_*\pi$, we use (3.2.10) and the fact that $\pi(h(0))\Upsilon = \langle \lambda, h\rangle\Upsilon$ whenever $h \in \mathfrak{t}_\mathbb{C}$

$$\zeta_*\pi(h(0))\Omega = \langle h, w_j\lambda + \ell\lambda_j^\vee\rangle\Omega = \langle h, \zeta\lambda\rangle\Omega \tag{3.2.13}$$

$\diamond$

We now derive a number of simple corollaries of the above results. First, the level of a positive energy representation may be detected globally in view of the following

Corollary 3.2.4. *Let $(\pi, \mathcal{H})$ be a level ℓ positive energy representation of LG. Then, for any $\tau \in T$ and coroot $\alpha \in \Lambda_R^\vee$*

$$\pi(\zeta_\alpha)\pi(\tau)\pi(\zeta_\alpha)^*\pi(\tau)^* = \alpha(\tau)^{-\ell} \tag{3.2.14}$$

where $\zeta_\alpha(\phi) = \exp_T(-i\alpha\phi)$ and the right hand–side of (3.2.14) *refers to the canonical pairing $\Lambda_R^\vee \times T \to \mathbb{T}$, $(\alpha, \exp_T(h)) \to e^{\langle \alpha, h\rangle}$.*

Proof. Since ζ_α and T commute in LG, the following holds in $PU(\mathcal{H})$ for any $h \in \mathfrak{t}$ and $s \in \mathbb{R}$

$$\pi(\zeta_\alpha)e^{s\pi(h)}\pi(\zeta_\alpha)^* = \pi(\zeta_\alpha)\pi(\exp_T(sh))\pi(\zeta_\alpha)^* = \pi(\exp_T(sh)) = e^{s\pi(h)} \tag{3.2.15}$$

Thus, in $U(\mathcal{H})$,

$$\pi(\zeta_\alpha)e^{s\pi(h)}\pi(\zeta_\alpha)^* = \lambda(s)e^{s\pi(h)} \tag{3.2.16}$$

where $\lambda : \mathbb{R} \to \mathbb{T}$ is a continuous homomorphism and is therefore of the form e^{sc} for some $c \in i\mathbb{R}$. Let now $\xi \in \mathcal{H}^{\text{fin}}$. By (ii) of proposition 3.2.2, $\pi(\zeta_\alpha)^*\xi \in \mathcal{H}^{\text{fin}}$ so that applying both sides of (3.2.16) to ξ and differentiating at $s = 0$, we find

$$\pi(\zeta_\alpha)\pi(h)\pi(\zeta_\alpha)^*\xi = \pi(h)\xi + c\xi \tag{3.2.17}$$

The result now follows by comparison with (iii) of proposition 3.2.2. Indeed, $\zeta_\alpha \in LG$ and therefore ${\zeta_\alpha^{-1}}_*\pi(X) = \pi(\zeta_\alpha)\pi(X)\pi(\zeta_\alpha)^*$ for any $X \in L^{\text{pol}}\mathfrak{g}$. Since $(\dot{\zeta_\alpha^{-1}})\zeta_\alpha = i\alpha$, we find by equating (3.2.17) and (3.2.3), that $c = -\ell\langle \alpha, h\rangle$ $\diamond$

Corollary 3.2.5. *If G is simply–laced, the action of $Z(G)$ via conjugation by discontinuous loops on the irreducible level 1 positive energy representations of LG is transitive and free.*

Proof. This follows at once from theorem 3.2.3 and corollary 3.1.6 $\diamond$

Corollary 3.2.6. *If ℓ is odd, the action of each of the two elements of $Z(\mathrm{Spin}_{2n})$ corresponding to $-1 \in Z(\mathrm{SO}_{2n})$ on the positive energy representations of level ℓ maps those whose lowest energy subspace is a single–valued SO_{2n}–module to those whose lowest energy subspace is a two–valued SO_{2n}–module and viceversa.*

Proof. This follows from proposition 3.2.3 and inspection of the tables in §3.3 as follows. The highest weights of irreducible Spin_{2n}–modules are given by sequences $\mu_1 \geq \cdots \geq \mu_{n-1} \geq |\mu_n|$ where the μ_j are either all integral or half–integral, the latter being the case iff the corresponding representation is a two–valued SO_{2n}–module. The elements of $Z(\mathrm{Spin}_{2n})$ mapping to $-1 \in \mathrm{SO}_{2n}$ correspond to the fundamental (co)weights $\lambda_{n-1}^\vee$ and $\lambda_n^\vee$ whose associated representations are the spin modules. The result now follows from equations (3.3.5)–(3.3.6) $\diamond$

3.3. Appendix : explicit action of $Z(G)$ on the level ℓ alcove.

We give the explicit action of $Z(G)$ on $\mathcal{A}_\ell$ for each G. For simply–laced G, the coroot and coweight lattices are identified with the root and weight lattices respectively. We denote by θ_i, $i = 1 \dots n$ and $\langle \cdot, \cdot \rangle$ the standard basis and inner product in $\mathbb{R}^n$, by I the self-dual lattice $\bigoplus_i \theta_i \mathbb{Z}$ and by $I^0 = \{\lambda \in I | \, |\lambda| = \sum_i \lambda_i \in 2\mathbb{Z}\}$. Unless otherwise indicated, the basic inner product is the standard one.

$\mathbf{SU_n}$, $\mathbf{n \geq 2}$ (simply-laced)
roots : $\theta_i - \theta_j$, $i \neq j$.
root lattice : $\Lambda_R = \{\xi \in I | \sum_i \xi_i = 0\}$.
simple roots : $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1 \dots n-1$.
highest root : $\theta = \theta_1 - \theta_n = \alpha_1 + \cdots + \alpha_{n-1}$.
fundamental weights : $\lambda_i = \theta_1 + \cdots + \theta_i - \frac{i}{n} \sum_j \theta_j$.
weight lattice : generated by the λ_i but more conveniently identified with $I/\xi\mathbb{Z}$ where $\xi = \sum \theta_j$ with inner product $\langle \lambda, \mu \rangle = \langle \lambda - \frac{\langle \lambda, \xi \rangle}{\langle \xi, \xi \rangle} \xi, \mu - \frac{\langle \mu, \xi \rangle}{\langle \xi, \xi \rangle} \xi \rangle$.
centre : $\lambda_k = k\lambda_1 \bmod \Lambda_R$ and therefore $\Lambda_W / \Lambda_R \cong \mathbb{Z}_n$ is generated by λ_1.
Weyl group : $\mathfrak{S}_n$ acting by permutation of the θ_i.
W_0 : w_k is the cyclic permutation $(\theta_1 \cdots \theta_n)^k = (\alpha_0 \cdots \alpha_{n-1})^k$.
level ℓ alcove : $\mathcal{A}_\ell = \{\lambda \in I | \, \lambda_1 \geq \cdots \geq \lambda_n, \; \lambda_1 - \lambda_n \leq \ell\} / (\sum_j \theta_j)$.
action of the centre : $A_k(\mu_1, \dots, \mu_n) = (\ell + \mu_{n+1-k}, \dots, \ell + \mu_n, \mu_1, \dots, \mu_{n-k})$.

$\mathbf{Spin_{2n+1}}$
Since $\mathrm{Spin}_3 \cong \mathrm{SU}_2$, we assume $n \geq 2$.
roots : $\pm\theta_i \pm \theta_j$, $i \neq j$ and $\pm\theta_i$.
root lattice : $\Lambda_R = I$.
simple roots : $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1 \dots n-1$ and $\alpha_n = \theta_n$.
highest root : $\theta = \theta_1 + \theta_2 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$.
coroots : $\pm\theta_i \pm \theta_j$, $i \neq j$ and $\pm 2\theta_i$.
coroot lattice : $\Lambda_R^\vee = I^0$.
simple coroots : $\alpha_i^\vee = \theta_i - \theta_{i+1}$, $i = 1 \dots n-1$ and $\alpha_n^\vee = 2\theta_n$.
highest coroot : $\widehat{\theta} = 2\theta_1 = 2(\alpha_1^\vee + \cdots + \alpha_{n-1}^\vee) + \alpha_n^\vee$.
coweight lattice : $\Lambda_W^\vee = I^* = I$.
fundamental coweights : $\lambda_i^\vee = \theta_1 + \cdots + \theta_i$, $i = 1 \dots n$.
centre : $\Lambda_W^\vee / \Lambda_R^\vee \cong \mathbb{Z}_2$ generated by $\lambda_1^\vee$.
weight lattice : $\Lambda_W = {\Lambda_R^\vee}^* = I + \frac{1}{2}(\theta_1 + \cdots + \theta_n)\mathbb{Z}$.
fundamental weights : $\lambda_i = \theta_1 + \cdots + \theta_i$, $i = 1 \dots n-1$ and $\lambda_n = \frac{1}{2}(\theta_1 + \cdots + \theta_n)$.
dual of centre : $\Lambda_W / \Lambda_R \cong \mathbb{Z}_2$ generated by λ_n.
Weyl group : $\mathfrak{S}_n \ltimes \mathbb{Z}_2^n$ acting by permutations and sign changes of the θ_i.
W_0 : w_1 is the sign change $\theta_1 \to -\theta_1$ permuting α_0 and α_1.
level ℓ alcove : $\mathcal{A}_\ell = \{\mu \in I + \frac{1}{2}(\theta_1 + \ldots + \theta_n)\mathbb{Z} | \, \mu_1 \geq \cdots \geq \mu_n \geq 0, \; \mu_1 + \mu_2 \leq \ell\}$.
action of the centre : $A_1(\mu_1, \dots, \mu_n) = (\ell - \mu_1, \mu_2, \dots, \mu_n)$.
level 1 weights : λ_1, λ_n.

$\mathbf{Sp_n}$
Since $\mathrm{Sp}_1 \cong \mathrm{SU}_2$, we assume $n \geq 2$.
roots : $\pm\theta_i \pm \theta_j$, $i \neq j$ and $\pm 2\theta_i$.
root lattice : $\Lambda_R = I^0$.
simple roots : $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1 \dots n-1$ and $\alpha_n = 2\theta_n$.
highest root : $\theta = 2\theta_1 = 2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n$.
basic inner product : half the standard one on $\mathbb{R}^n$.
coroots : $\pm 2(\theta_i \pm \theta_j)$, $i \neq j$ and $\pm 2\theta_i$.
coroot lattice : $\Lambda_R^\vee = 2I$.

simple coroots : $\alpha_i^\vee = 2(\theta_i - \theta_{i+1})$, $i = 1 \dots n-1$ and $\alpha_n^\vee = 2\theta_n$.
highest coroot : $\widehat{\theta} = 2(\theta_1 + \theta_2) = \alpha_1^\vee + 2(\alpha_2^\vee + \cdots + \alpha_n^\vee)$.
coweight lattice : $\Lambda_W^\vee = 2I + (\theta_1 + \cdots + \theta_n)\mathbb{Z}$.
fundamental coweights : $\lambda_i^\vee = 2(\theta_1 + \cdots + \theta_i)$, $i = 1 \dots n-1$ and $\lambda_n^\vee = \theta_1 + \ldots + \theta_n$.
centre : $\Lambda_W^\vee/\Lambda_R^\vee \cong \mathbb{Z}_2$ generated by $\lambda_n^\vee$.
weight lattice : $\Lambda_W = I$.
fundamental weights : $\lambda_i = \theta_1 + \cdots + \theta_i$, $i = 1 \dots n$.
dual of centre : $\Lambda_W/\Lambda_R \cong \mathbb{Z}_2$ generated by λ_1.
Weyl group : $\mathfrak{S}_n \ltimes \mathbb{Z}_2^n$ acting by permutations and sign changes of the θ_i.
W_0 : w_n is the transformation $\theta_i \to -\theta_{n+1-i}$.
level ℓ alcove : $\mathcal{A}_\ell = \{\mu \in I |\, \ell \geq \mu_1 \geq \cdots \geq \mu_n \geq 0\}$.
action of the centre : $A_n(\mu_1, \dots, \mu_n) = (\ell - \mu_n, \dots, \ell - \mu_1)$.
level 1 weights : λ_i, $i = 1 \dots n$.

$\mathbf{Spin_{2n}}$, $\mathbf{n \geq 3}$ (simply laced)
roots : $\pm\theta_i \pm \theta_j$, $i \neq j$.
root lattice : $\Lambda_R = \Lambda_R^\vee = I^0$.
simple roots : $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1 \dots n-1$ and $\alpha_n = \theta_{n-1} + \theta_n$.
highest root : $\theta = \theta_1 + \theta_2 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$.
weight lattice : $\Lambda_W = \Lambda_W^\vee = I + \frac{1}{2}(\theta_1 + \cdots + \theta_n)\mathbb{Z}$.
fundamental weights :

$$\lambda_i = \theta_1 + \cdots + \theta_i \qquad i = 1 \dots n-2 \tag{3.3.1}$$

$$\lambda_{n-1} = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} - \theta_n) \tag{3.3.2}$$

$$\lambda_n = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} + \theta_n) \tag{3.3.3}$$

centre : We have $2\lambda_1 = 2\theta_1 \in \Lambda_R$. Moreover, for n even, $2\lambda_{n-1} = (\theta_1 + \theta_2) + \cdots + (\theta_{n-1} \pm \theta_n) = 0 \bmod \Lambda_R$ and $2\lambda_n = 0 \bmod \Lambda_R$. On the other hand, for n odd, $2\lambda_n = \theta_1 + (\theta_2 + \theta_3) + \cdots + (\theta_{n-1} \pm \theta_n) = \lambda_1 \bmod \Lambda_R$ and similarly $2\lambda_{n-1} = \lambda_1 \bmod \Lambda_R$. Thus, $Z(\mathrm{Spin}_{2n})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for n even and to $\mathbb{Z}_4$ for n odd with λ_{n-1} and λ_n of order 4.
Weyl group : $\mathfrak{S}_n \ltimes \mathbb{Z}_2^{n-1}$ acting by permutations and even numbers of sign changes of the θ_i.
W_0 : w_1 is the sign change $\theta_1 \to -\theta_1$, $\theta_n \to -\theta_n$ and permutes $\{\alpha_0, \alpha_1\}$ and $\{\alpha_{n-1}, \alpha_n\}$. For n even, w_{n-1} is given by $\theta_i \to -\theta_{n+1-i}$, $2 \leq i \leq n-1$ and $\theta_1 \leftrightarrow \theta_n$ and permutes $\{\alpha_0, \alpha_{n-1}\}$ and $\{\alpha_1, \alpha_n\}$ while w_n is given by $\theta_i \to -\theta_{n+1-i}$ and permutes $\{\alpha_k, \alpha_{n-k}\}$. For n odd, w_{n-1} is given by $\theta_1 \to \theta_n$ and $\theta_i \to -\theta_{n+1-i}$, $i = 2 \dots n$ and acts as the cyclic permutation $\begin{pmatrix}\alpha_1 & \alpha_n & \alpha_0 & \alpha_{n-1}\end{pmatrix}$ while w_n is given by $\theta_i \to -\theta_{n+1-i}$, $i = 1 \dots n-1$ and $\theta_n \to \theta_1$ and acts as $\begin{pmatrix}\alpha_1 & \alpha_n & \alpha_0 & \alpha_{n-1}\end{pmatrix}^{-1}$.
level ℓ alcove : $\mathcal{A}_\ell = \{\mu \in I + \frac{1}{2}(\theta_1 + \ldots + \theta_n)\mathbb{Z} |\, \mu_1 \geq \cdots \geq \mu_{n-1} \geq |\mu_n|,\ \mu_1 + \mu_2 \leq \ell\}$.
action of the centre :

$$A_1(\mu_1, \dots, \mu_n) = (\ell - \mu_1, \mu_2, \dots, \mu_{n-1}, -\mu_n). \tag{3.3.4}$$

$$A_{n-1}(\mu_1, \dots, \mu_n) = \begin{cases} (\frac{\ell}{2} + \mu_n, \frac{\ell}{2} - \mu_{n-1}, \dots, \frac{\ell}{2} - \mu_2, -\frac{\ell}{2} + \mu_1) & n \text{ even} \\ (\frac{\ell}{2} - \mu_n, \dots, \frac{\ell}{2} - \mu_2, -\frac{\ell}{2} + \mu_1) & n \text{ odd} \end{cases} \tag{3.3.5}$$

$$A_n(\mu_1, \dots, \mu_n) = \begin{cases} (\frac{\ell}{2} - \mu_n, \dots, \frac{\ell}{2} - \mu_1) & n \text{ even} \\ (\frac{\ell}{2} + \mu_n, \frac{\ell}{2} - \mu_{n-1}, \dots, \frac{\ell}{2} - \mu_1) & n \text{ odd} \end{cases} \tag{3.3.6}$$

4. Definition and classification of primary fields

For any G–module V, $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ acts on the space of V–valued Laurent polyomials $V[z, z^{-1}]$ by multiplication. We extend this to an action of $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}} \rtimes \mathrm{Rot}\, S^1$ by setting $R_\theta f(z) = f(e^{-i\theta}z)$. In terms of the

basis $v(n) = v \otimes z^n$ of $V[z, z^{-1}]$, the action is given by

$$X(m)v(n) = Xv(n+m) \qquad\qquad dv(n) = -nv(n) \tag{4.1}$$

Let now $\ell \in \mathbb{N}$ and consider the set $\mathcal{P}_\ell$ of positive energy representations of LG at level ℓ.

Definition. Let $\mathcal{H}_i, \mathcal{H}_j \in \mathcal{P}_\ell$ be irreducibles and V_i, V_j the corresponding lowest energy subspaces. If V_k is an irreducible G–module, a *primary field* of charge V_k is a linear map $\phi : \mathcal{H}_i^{\mathrm{fin}} \otimes V_k[z, z^{-1}] \to \mathcal{H}_j^{\mathrm{fin}}$ intertwining the action of $L^{\mathrm{pol}}\mathfrak{g} \rtimes \mathrm{Rot}\, S^1$, so that

$$[X, \phi(f)] = \phi(Xf) \qquad\qquad [d, \phi(f)] = i\phi(\dot{f}) \tag{4.2}$$

We represent ϕ by the (formal) operator–valued distribution $\phi(v, z) = \sum_{n\in\mathbb{Z}} \phi(v, n) z^{-n-\Delta_\phi}$ where $\phi(v, n) = \phi(v \otimes z^n) : \mathcal{H}_i^{\mathrm{fin}} \to \mathcal{H}_j^{\mathrm{fin}}$. Here, $\Delta_\phi = \Delta_i + \Delta_k - \Delta_j$ is the *conformal weight* of ϕ, the summands being the Casimirs of the corresponding G-modules divided by 2κ where $\kappa = \ell + \frac{C_\mathfrak{g}}{2}$ and $C_\mathfrak{g}$ is the Casimir of the adjoint representation. The intertwining properties of ϕ may then be rephrased as

$$[X(m), \phi(v, n)] = \phi(Xv, m+n) \qquad\qquad [d, \phi(v, n)] = -n\phi(v, n) \tag{4.3}$$

or as

$$[X(m), \phi(v, z)] = \phi(Xv, z)z^m \qquad\qquad [d, \phi(v, z)] = (z\frac{d}{dz} + \Delta_\phi)\phi(v, z) \tag{4.4}$$

By restriction, $\phi(\cdot, 0)$ gives rise to a G-intertwiner or *initial term* $\varphi : V_i \otimes V_k \to V_j$ which determines ϕ uniquely. In fact, the description of $\mathcal{H}_i^{\mathrm{fin}}$ as the quotient of the module freely generated by the action of $\widehat{\mathfrak{g}}_- = \oplus_{n<0} \mathfrak{g}_\mathbb{C} \otimes z^n$ on V_i modulo the single relation $e_\theta(-1)^{\ell - \langle\lambda_i, \theta\rangle + 1} v_i = 0$ where λ_i is the highest weight of V_i, v_i the corresponding eigenvector and θ the highest root of G [**Ka1**], entails the following

Proposition 4.1 (Tsuchiya–Kanie). *There exists a (necessarily unique) primary field*

$$\phi : \mathcal{H}_i^{\mathrm{fin}} \otimes V_k[z, z^{-1}] \to \mathcal{H}_j^{\mathrm{fin}} \tag{4.5}$$

with given initial term $\varphi \in \mathrm{Hom}_G(V_i \otimes V_k, V_j)$ if, and only if the restriction of φ to any triple $U_i \subset V_i$, $U_k \subset V_k$, $U_j \subset V_j$ of irreducible $\{e_\theta, f_\theta, h_\theta\}$–submodules with highest weights s_i, s_k, s_j is zero whenever $s_i + s_k + s_j \geq 2\ell + 1$. If φ is non–zero and satisfies this criterion, then V_k is admissible at level ℓ. In particular, the charge of a non–zero primary field is necessarily ℓ-admissible.

Proof. The above is proved in [**TK1**, thm. 2.3] for $G = \mathrm{SU}_2$ and in [**Wa2**] for $G = \mathrm{SU}_n$. The extension to any G is immediate $\diamond$

Remark. Following Tsuchiya and Kanie, we shall represent the G–modules attached to a primary field $\phi : \mathcal{H}_i^{\mathrm{fin}} \otimes V_k[z, z^{-1}] \to \mathcal{H}_j^{\mathrm{fin}}$ by the *vertex* $\begin{pmatrix} & V_k & \\ V_j & & V_i \end{pmatrix}$ or, more succintly $\begin{pmatrix} & k & \\ j & & i \end{pmatrix}$.

Definition. For any triple V_i, V_k, V_j of irreducible representations of G, let $\mathrm{Hom}_G^\ell(V_i \otimes V_k, V_j) \subset \mathrm{Hom}_G(V_i \otimes V_k, V_j)$ be the subspace of intertwiners satisfying the condition of proposition 4.1. By symmetry of the latter, the isomorphisms $\mathrm{Hom}_G(V_i \otimes V_k, V_j) \cong \mathrm{Hom}_G(V_k \otimes V_i, V_j)$ and $\mathrm{Hom}_G(V_i \otimes V_k, V_j) \cong \mathrm{Hom}_G(V_j^* \otimes V_k, V_i^*)$ restrict to isomorphisms of the corresponding Hom_G^ℓ spaces and the same is true for any permutation of V_i, V_j, V_k.

Lemma 4.2. *Let V_i, V_k, V_j be irreducible G-modules which are admissible at level ℓ. If one of them is minimal, then $\mathrm{Hom}_G^\ell(V_i \otimes V_k, V_j) = \mathrm{Hom}_G(V_i \otimes V_k, V_j)$.*

Proof. Up to a permutation, we may assume that V_k is minimal. By corollary 2.2.3, $\mathrm{Hom}_G(V_i \otimes V_k, V_j)$ is at most one-dimensional. Assume the generator φ restricts to a non–zero intertwiner $U_i \otimes U_k \to U_j$ for some triple of $\{e_\theta, f_\theta, h_\theta\}$–submodules with highest weights s_i, s_k, s_j. Since V_k is minimal, $s_k \in \{0, 1\}$ and therefore, by the Clebsch-Gordan rules $\min(s_i, s_j) = \max(s_i, s_j) - s_k$ so that $s_i + s_j + s_k = 2\max(s_i, s_j) \leq 2\ell$ $\diamond$

DEFINITION. Let $\phi : \mathcal{H}_i^{\mathrm{fin}} \otimes V_k[z,z^{-1}] \to \mathcal{H}_j^{\mathrm{fin}}$ be a primary field with charge V_k. The *adjoint field* $\phi^* : \mathcal{H}_j^{\mathrm{fin}} \otimes V_k^*[z,z^{-1}] \to \mathcal{H}_i^{\mathrm{fin}}$ is the unique primary field with charge V_k^* satisfying

$$(\phi(f)\xi, \eta) = (\xi, \phi^*(\overline{f})\eta) \tag{4.6}$$

for any $\xi \in \mathcal{H}_i^{\mathrm{fin}}, \eta \in \mathcal{H}_j^{\mathrm{fin}}$ and $f \in V_k[z,z^{-1}]$. It is defined in the following way. Using the anti–linear identification $\bar{\cdot} : V_k \to V_k^*$, set

$$\phi^*(\overline{v}, n) = \phi(v, -n)^* \tag{4.7}$$

then ϕ^* satisfies (4.6) and defines the required primary field.

CHAPTER II

Analytic properties of positive energy representations

Using the Segal–Sugawara formula, we show in section 1 that the infinitesimal action of $L^{\mathrm{pol}}\mathfrak{g}$ on the finite energy subspace of a positive energy representation $\mathcal{H}$ extends to one of $L\mathfrak{g}$ on the space $\mathcal{H}^\infty$ of *smooth vectors* for $\mathrm{Rot}\,S^1$. When exponentiated, this action coincides with the original representation of LG. We prove that $\mathcal{H}^\infty$ is invariant under LG and find that the crossed homomorphism for the joint projective action of LG and $L\mathfrak{g}\rtimes i\mathbb{R}d$ on $\mathcal{H}^\infty$ agrees with that given by Pressley and Segal [**PS**, 4.9.4].

In section 2, we show that the topological central extensions of LG arising from positive energy representations are smooth, in fact real–analytic and compute their Lie algebra cocycle. Their classification follows at once from that of Pressley and Segal [**PS**, 4.4.1] and shows in particular that the central extensions corresponding to positive energy representations of the same level are canonically isomorphic. Moreover, every positive energy representation possesses a dense subspace of analytic, and *a fortiori* smooth vectors for LG, a result conjectured by Pressley and Segal [**PS**, §9.3].

1. The subspace of smooth vectors

Let $(\pi,\mathcal{H})$ be a positive energy representation of LG and d the self–adjoint generator of rotations normalised by $d|_{\mathcal{H}(n)}=n$, $n\in\mathbb{N}$. As pointed out by Goodman and Wallach [**GoWa**], the Segal–Sugawara formula ([**PS**, §9.4] and §1.2 below) implies that d plays a rôle analogous to that of the laplacian of LG in the representation. In particular, we shall prove in §1.5 that the action of LG preserves the abstract Sobolev scale corresponding to $\mathrm{Rot}\,S^1$ which we presently define. Let $\|\cdot\|_s$, $s\in\mathbb{R}$ be the Sobolev norm on $\mathcal{H}^{\mathrm{fin}}$ given by

$$\|\xi\|_s^2=\|(1+d)^s\xi\|^2=((1+d)^{2s}\xi,\xi) \tag{1.1}$$

where the powers $(1+d)^s$ are defined by the spectral theorem or simply by expanding ξ as a sum of eigenvectors of d. The *scale* $\mathcal{H}^s$ is the Hilbert space completion of $\mathcal{H}^{\mathrm{fin}}$ with respect to $\|\cdot\|_s$. Thus, $\mathcal{H}^s=\mathcal{D}((1+d)^s)$ for $s\geq 0$ and $(1+d)$ gives a unitary $\mathcal{H}^t\rightarrow\mathcal{H}^{t-1}$ for any t. The space $\mathcal{H}^\infty$ of *smooth vectors* for $\mathrm{Rot}\,S^1$ is, by definition, $\bigcap_s\mathcal{H}^s$. When endowed with the corresponding direct limit topology, $\mathcal{H}^\infty$ is a Fréchet space which may equivalently be described as the space of $\xi\in\mathcal{H}$ such that the function $\theta\rightarrow\pi(R_\theta)\xi$ is smooth, topologised as a closed subspace of $C^\infty(S^1,\mathcal{H})$.

1.1. The Banach Lie algebras $L\mathfrak{g}_t$.

We shall need an alternative description of the Fréchet topology on $L\mathfrak{g}$. The restriction of the basic inner product $\langle\cdot,\cdot\rangle$ to $i\mathfrak{g}\subset\mathfrak{g}_{\mathbb{C}}$ is positive definite and we extend it to a G-invariant hermitian form on $\mathfrak{g}_{\mathbb{C}}$ with associated norm $\|\cdot\|$. For $X=\sum_k a_k e^{ik\theta}\in L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ and $t\geq 0$, define

$$|X|_t=\sum_k(1+|k|)^t\|a_k\| \tag{1.1.1}$$

Let $L\mathfrak{g}_t$ be the completion of $L^{\mathrm{pol}}\mathfrak{g}$ with respect to $|\cdot|_t$. If a_k, b_l are the Fourier coefficents of $X,Y\in L^{\mathrm{pol}}\mathfrak{g}$ and $C_{\mathfrak{g}}>0$ is such that $\|[x,y]\|\leq C_{\mathfrak{g}}\|x\|\|y\|$ for any $x,y\in\mathfrak{g}$, then

$$|[X,Y]|_t\leq C_{\mathfrak{g}}\sum_{k,l}\|a_k\|\|b_l\|(1+|k+l|)^t\leq C_{\mathfrak{g}}|X|_t|Y|_t \tag{1.1.2}$$

since $(1+|k+l|)^t \leq (1+|k|)^t(1+|l|)^t$ for $t \geq 0$. $L\mathfrak{g}_t$ is therefore a Banach Lie algebra. If $\|.\|_\infty$ is the supremum norm on $C(S^1,\mathfrak{g})$, and $X=\sum_k a_k e^{ik\theta} \in L^{\mathrm{pol}}\mathfrak{g}$, then for any $t \geq n$

$$\|X^{(n)}\|_\infty \leq \sum_k \|a_k\||k|^n \leq |X|_t \tag{1.1.3}$$

Moreover, if $2 \geq s > 1$,

$$\begin{aligned} |X|_t &= \sum_k (1+|k|)^{-\frac{s}{2}}(1+|k|)^{t+\frac{s}{2}}\|a_k\| \\ &\leq \Big\{\sum_k (1+|k|)^{-s}\Big\}^{\frac{1}{2}}\Big\{\sum_k (1+|k|)^{2t+s}\|a_k\|^2\Big\}^{\frac{1}{2}} \\ &= C_s\|(1+|\frac{d}{d\theta}|)^{t+\frac{s}{2}}X\|_{L^2(S^1,\mathfrak{g})} \\ &\leq C_s'\|X\|_{C^{\lceil t\rceil+1}(S^1,\mathfrak{g})} \end{aligned} \tag{1.1.4}$$

Consequently, we have the following norm-continuous embeddings with dense image

$$C^{\lceil t\rceil+1}(S^1,\mathfrak{g}) \hookrightarrow L\mathfrak{g}_t \hookrightarrow C^{\lfloor t\rfloor}(S^1,\mathfrak{g}) \tag{1.1.5}$$

showing that the Fréchet topology on $L\mathfrak{g}$ given by the $C^k(S^1,\mathfrak{g})$ norms may equivalently be given by the norms $|\cdot|_t$.

1.2. The Segal–Sugawara formula and Sobolev estimates on the action of $L^{\mathrm{pol}}\mathfrak{g}$.

Let $\mathcal{H}$ be a positive energy representation of LG at level ℓ. The restriction of the infinitesimal generator of rotations to $\mathcal{H}^{\mathrm{fin}}$ is given via the Segal–Sugawara formula [**PS**, §9.4]

$$L_0 = \frac{1}{\kappa}\Big(\frac{1}{2}X_i(0)X^i(0) + \sum_{m>0} X_i(-m)X^i(m)\Big) \tag{1.2.1}$$

where $\{X_i\},\{X^i\}$ are dual basis of $\mathfrak{g}_{\mathbb{C}}$ with respect to the basic inner product and $\kappa = \ell + g$ with g the dual Coxeter number of $\mathfrak{g}$ (half the Casimir of the adjoint representation). More precisely, L_0 satisfies $[L_0, x(n)] = -nx(n)$ and therefore differs from d by an additive constant on each irreducible summand of $\mathcal{H}^{\mathrm{fin}}$. If $\mathcal{H}$, and therefore $\mathcal{H}(0)$, is irreducible, L_0 acts on $\mathcal{H}(0)$ as the scalar $\Delta = \frac{C}{2\kappa}$, where C is the Casimir of $\mathcal{H}(0)$ so that

$$L_0 = d + \Delta \tag{1.2.2}$$

More generally, only finitely many irreducible G-modules appear as lowest energy subspaces of positive energy representations at level ℓ. It follows that for any level ℓ representation $\mathcal{H}$, $d \leq L_0 \leq d + C_\ell$ for some constant C_ℓ depending only on ℓ and therefore that d and L_0 define the same scales. For convenience, we shall often supersede definition (1.1) and refer to $\|(1+L_0)^s\xi\|$ as $\|\xi\|_s$. The following estimates are due to Goodman and Wallach [**GoWa**, §3.2]

PROPOSITION 1.2.1. *Let $X \in L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$ and $\xi \in \mathcal{H}^{\mathrm{fin}}$. Then, for any $s \in \mathbb{R}$*

$$\|\pi(X)\xi\|_s \leq \sqrt{2\kappa}|X|_{|s|+\frac{1}{2}}\|\xi\|_{s+\frac{1}{2}} \tag{1.2.3}$$

PROOF. It is clearly sufficient to show that for any $x \in \mathfrak{g}$ of norm 1 and $p \in \mathbb{Z}$

$$\|x(p)\xi\|_s^2 \leq 2\kappa(1+|p|)^{2|s|+1}\|\xi\|_{s+\frac{1}{2}}^2 \tag{1.2.4}$$

Let X_i be an orthonormal basis of $\mathfrak{g}$ so that X_i and $-X_i$ are dual basis with respect to $\langle\cdot,\cdot\rangle$. By formal adjunction, $X_i(n)^* = -X_i(-n)$ and therefore, by (1.2.1)

$$\begin{aligned} (L_0\xi,\xi) &= \frac{1}{2\kappa}\sum_i\Big(-(X_i(0)X_i(0)\xi,\xi) - 2\sum_{p>0}(X_i(-p)X_i(p)\xi,\xi)\Big) \\ &= \frac{1}{2\kappa}\sum_i\Big(\|X_i(0)\xi\|^2 + 2\sum_{p>0}\|X_i(p)\xi\|^2\Big) \end{aligned} \tag{1.2.5}$$

whence, for any $p \geq 0$, $\|X_i(p)\xi\|^2 \leq 2\kappa\|L_0^{\frac{1}{2}}\xi\|^2$. If $p < 0$, $[X_i(p), X_i(-p)] = -p$ and therefore $\|X_i(p)\xi\|^2 = \|X_i(-p)\xi\|^2 - p\|\xi\|^2$. Thus, for any p

$$\|X_i(p)\xi\|^2 \leq 2\kappa\|L_0^{\frac{1}{2}}\xi\|^2 + |p|\|\xi\|^2 \tag{1.2.6}$$

Since the eigenspaces of L_0 are orthonormal for any of the Sobolev norms, is it sufficient, to show (1.2.4), to restrict to the case $L_0\xi = m\xi$. Therefore, assuming $p \leq m$ (else the left hand side vanishes),

$$\begin{aligned}\|X_i(p)\xi\|_s^2 &= \|(1+L_0)^s X_i(p)\xi\|^2 \\ &= (1+m-p)^{2s}\|X_i(p)\xi\|^2 \\ &\leq (1+m-p)^{2s}(2\kappa m + |p|)\|\xi\|^2 \\ &\leq 2\kappa\frac{(1+m-p)^{2s}}{(1+m)^{2s}}(1+|p|)\|\xi\|^2_{s+\frac{1}{2}}\end{aligned} \tag{1.2.7}$$

If $s \geq 0$, $\frac{(1+m-p)^{2s}}{(1+m)^{2s}} \leq g(m) = \frac{(1+m+|p|)^{2s}}{(1+m)^{2s}}$ and the bound is decreasing in m so that $g(m) \leq g(0) = (1+|p|)^{2s}$. If, on the contrary, $s < 0$ and $p \leq 0$ we have $\frac{(1+m-p)^{2s}}{(1+m)^{2s}} = (1 - \frac{p}{m+1})^{2s} \leq 1$. Lastly, if $0 < p \geq m$, $h(m) = \frac{(1+m-p)^{2s}}{(1+m)^{2s}}$ is decreasing in m and therefore $h(m) \leq h(p) = (1+p)^{2|s|}$. Either way, we obtain

$$\|X_i(p)\xi\|_s^2 \leq 2\kappa(1+|p|)^{2|s|+1}\|\xi\|^2_{s+\frac{1}{2}} \tag{1.2.8}$$

$\diamond$

1.3. The action of $L\mathfrak{g}$ on $\mathcal{H}^\infty$.

Let $(\pi, \mathcal{H})$ be a level ℓ positive energy representation of LG with finite energy subspace $\mathcal{H}^{\text{fin}}$. Then,

COROLLARY 1.3.1. *The action of $L^{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ on $\mathcal{H}^{\text{fin}}$ given by theorem I.1.2.1 extends to a jointly continuous, projective action $L\mathfrak{g}_{\mathbb{C}} \otimes \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$ satisfying*

$$[d, \pi(X)] = i\pi(\dot{X}) \tag{1.3.1}$$

$$[\pi(X), \pi(Y)] = \pi([X,Y]) + i\ell B(X,Y) \tag{1.3.2}$$

where, as customary $B(X,Y) = \int_0^{2\pi}\langle X, \dot{Y}\rangle\frac{d\theta}{2\pi}$. Moreover, the operators $\pi(X)$, $X \in L\mathfrak{g}$ are essentially skew–adjoint on $\mathcal{H}^\infty$.

PROOF. The first claim is an immediate consequence of the estimates of proposition 1.2.1. The second follows from the estimates

$$\|\pi(X)\xi\|_s \leq \sqrt{2\kappa}|X|_{|s|+\frac{1}{2}}\|\xi\|_{s+\frac{1}{2}} \tag{1.3.3}$$

$$\|[1+d, \pi(X)]\xi\|_s \leq \sqrt{2\kappa}|\dot{X}|_{|s|+\frac{1}{2}}\|\xi\|_{s+\frac{1}{2}} \leq \sqrt{2\kappa}|X|_{|s|+\frac{3}{2}}\|\xi\|_{s+\frac{1}{2}} \tag{1.3.4}$$

and Nelson's commutator theorem [**Ne2**] $\diamond$

Clearly, π extends to a jointly continuous map $L\mathfrak{g}_{\mathbb{C}} \otimes \mathcal{H}^s \rightarrow \mathcal{H}^{s-\frac{1}{2}}$. It follows that the restriction of the operators $\pi(X)$, $X \in L\mathfrak{g}$ is essentially skew–adjoint on any dense subspace of $\mathcal{H}^\infty$ for the Fréchet topology by Nelson's commutator theorem [**Ne2**]. We shall usually consider the operators $\pi(X)$ as being defined on the invariant core $\mathcal{H}^\infty$. When no confusion arises, we denote by the same symbol their restriction to any of the $\mathcal{H}^s$ and their skew–adjoint closure.

The following shows that the exponentiation of the representation of $L\mathfrak{g}$ on $\mathcal{H}^\infty$ coincides with the original representation of LG.

PROPOSITION 1.3.2. *For any $X \in L\mathfrak{g}$, the following holds in $PU(\mathcal{H})$*

$$\pi(\exp_{LG} X) = e^{\pi(X)} \tag{1.3.5}$$

PROOF. When $X \in L^{\text{pol}}\mathfrak{g}$, (1.3.5) holds by the definition of the operators $\pi(X)$. Assume now $X \in L\mathfrak{g}$ and let $X_n \in L^{\text{pol}}\mathfrak{g}$, $X_n \to X$ in $L\mathfrak{g}$, and *a fortiori* in the $|\cdot|_{\frac{1}{2}}$ norm. By proposition 1.2.1,

$$\|(\pi(X) - \pi(X_n))\xi\| \leq \sqrt{2\kappa}|X - X_n|_{\frac{1}{2}}\|(1 + L_0)^{\frac{1}{2}}\xi\| \tag{1.3.6}$$

for any $\xi \in \mathcal{H}^\infty$. Since $\mathcal{H}^\infty$ is a core for $\pi(L\mathfrak{g})$, we deduce that $\pi(X_n) \to \pi(X)$ in the strong-resolvent sense [**RS**, thm. VIII.25 (a)] and therefore, by Trotter's theorem $e^{\pi(X_n)} \to e^{\pi(X)}$ strongly in $U(\mathcal{H})$ [**RS**, thm. VIII.21]. Thus,

$$e^{\pi(X)} = \lim_{n\to\infty} e^{\pi(X_n)} = \lim_{n\to\infty} \pi(\exp_{LG}(X_n)) = \pi(\exp_{LG}(X)) \tag{1.3.7}$$

$\diamond$

1.4. The exponential map of $LG \rtimes \mathrm{Rot}\, S^1$.

We study below the well–posedness of the exponential map of $LG \rtimes \mathrm{Rot}\, S^1$ and obtain an explicit formula for $\exp_{LG\rtimes\mathrm{Rot}\, S^1}(i\theta d + X)$, $X \in L\mathfrak{g}$. This will be needed to extend proposition 1.3.2 to $X \in L\mathfrak{g} \rtimes id\mathbb{R}$.

The integration of one–parameter groups in $LG \rtimes \mathrm{Rot}\, S^1$ reduces, after some simple manipulations to that of a linear, time–dependent ordinary differential equation for a function with values in LG. The latter is easily solved using Volterra product integrals whose salient features we now recall. Details may be found in [**Ne1**, chap. 2]. Let E be a Banach space and $\mathcal{B}(E)$ the Banach algebra of all bounded linear maps $E \to E$ endowed with the operator norm. For $A \in \mathcal{B}(E)$, define the operator e^A by the convergent power series $\sum_n \frac{A^n}{n!}$. Clearly, $\|e^A\| \leq \sum_n \frac{\|A\|^n}{n!} \leq e^{\|A\|}$. Moreover, from $A^n - B^n = \sum_{k=0}^{n-1} A^k(A - B)B^{n-1-k}$ we deduce that

$$\|e^A - e^B\| \leq \sum_{n\geq 0} \frac{\|A^n - B^n\|}{n!} \leq \|A - B\| \sum_{n\geq 1} \frac{\max(\|A\|, \|B\|)^{n-1}}{(n-1)!} = \|A - B\|e^{\max(\|A\|,\|B\|)} \tag{1.4.1}$$

For any $A \in C(\mathbb{R}, \mathcal{B}(E))$ and $a < b \in \mathbb{R}$, we define the product integral $\prod_{b\geq\tau\geq a} \mathrm{Exp}(A(\tau)d\tau)$ as follows. Consider first step functions $A : [a, b] \to \mathcal{B}(E)$, $A(\tau) = A_j$ for $\tau_j > \tau > \tau_{j-1}$ corresponding to subdivisions $b = \tau_n > \tau_{n-1} > \cdots > \tau_1 > \tau_0 = a$ and set

$$\prod_{b\geq\tau\geq a} \mathrm{Exp}(A(\tau)d\tau) := e^{\Delta_n A_n} \cdots e^{\Delta_1 A_1} \tag{1.4.2}$$

where $\Delta_j = \tau_j - \tau_{j-1}$. If A, B are two step functions, which we may take as defined on a common subdivision, the identity

$$e^{\Delta_n A_n} \cdots e^{\Delta_1 A_1} - e^{\Delta_n B_n} \cdots e^{\Delta_1 B_1} = \sum_{k=1}^{n} e^{\Delta_n A_n} \cdots e^{\Delta_{k+1} A_{k+1}}(e^{\Delta_k A_k} - e^{\Delta_k B_k})e^{\Delta_{k-1} B_{k-1}} \cdots e^{\Delta_1 B_1} \tag{1.4.3}$$

and (1.4.1) imply that

$$\| \prod_{b\geq\tau\geq a} \mathrm{Exp}(A(\tau)d\tau) - \prod_{b\geq\tau\geq a} \mathrm{Exp}(B(\tau)d\tau)\| \leq e^{(b-a)\max(\|A\|_{[a,b]},\|B\|_{[a,b]})}(b - a)\|A - B\|_{[a,b]} \tag{1.4.4}$$

where the norms on the right hand-side refer to the supremum norm on functions $[a, b] \to \mathcal{B}(E)$. We may therefore define the product integral for any $A \in C([a, b], \mathcal{B}(E))$ by using a sequence of approximating step functions. Notice that product integrals are invertible operators, in fact $\prod_{b\geq\tau\geq a} \mathrm{Exp}(A(\tau)d\tau)^{-1} = \prod_{b\geq\tau\geq a} \mathrm{Exp}(-\check{A}(\tau)d\tau)$ where $\check{A}(\tau) = A(a + b - \tau)$ so that we may define, for $a > b$, $\prod_{b\geq\tau\geq a} \mathrm{Exp}(A(\tau)d\tau) := \prod_{a\geq\tau\geq b} \mathrm{Exp}(A(\tau)d\tau)^{-1}$.

THEOREM 1.4.1. *Let E be a Banach space and $A : \mathbb{R} \to \mathcal{B}(E)$ a norm-continuous map. For any $\xi_0 \in E$, the time-dependent, linear ordinary differential equation in $\xi : \mathbb{R} \to E$*

$$\dot{\xi}(t) = A(t)\xi(t) \tag{1.4.5}$$

$$\xi(0) = \xi_0 \tag{1.4.6}$$

possesses a unique solution $\xi \in C^1(\mathbb{R}, E)$ given by

$$\xi(t) = \prod_{t \geq \tau \geq 0} \mathrm{Exp}(A(\tau) d\tau) \xi_0 \tag{1.4.7}$$

Moreover, if $A, \widetilde{A} \in C(\mathbb{R}, \mathcal{B}(E))$ and $\xi, \widetilde{\xi}$ are the corresponding solutions with initial condition ξ_0, then, for any $t \in \mathbb{R}$,

$$\|\xi(t) - \widetilde{\xi}(t)\| \leq e^{|t| \max(\|A\|_{[0,t]}, \|\widetilde{A}\|_{[0,t]})} |t| \|A - \widetilde{A}\|_{[0,t]} \|\xi_0\| \tag{1.4.8}$$

Proof. See [**Ne1**, Thm. 1, page 17] $\diamond$

Corollary 1.4.2 ([**Wa2**]). *The exponential map $L\mathfrak{g} \rtimes i\mathbb{R}d \to LG \rtimes \mathrm{Rot}\, S^1$ is well–defined and continuous. It is given by*

$$\begin{aligned} \exp_{LG \rtimes i\mathbb{R}}(X + i\alpha d) &= \prod_{1 \geq t \geq 0} \mathrm{Exp}(X_{\alpha(1-t)} dt) R_\alpha \\ &= \lim_{n\to\infty} \exp_{LG}\Big(\frac{X}{n}\Big) \exp_{LG}\Big(\frac{X_{\alpha \cdot 1/n}}{n}\Big) \cdots \exp_{LG}\Big(\frac{X_{\alpha\cdot(n-1)/n}}{n}\Big) R_\alpha \end{aligned} \tag{1.4.9}$$

Proof. To compute the exponential map, fix $X + i\alpha d \in L\mathfrak{g} \rtimes i\mathbb{R}d$ and consider $f = \gamma R_\phi : \mathbb{R} \to LG \rtimes \mathrm{Rot}\, S^1$ satisfying $\dot{f} = (X + i\alpha d) f$ and $f(0) = 1$. As a manifold, $LG \rtimes \mathrm{Rot}\, S^1$ is the product of the two factors and therefore $s \to \exp_{LG}(sX) R_{s\alpha}$ is an integral curve for $X + i\alpha d$ at 1. Thus

$$(X + i\alpha d) f = \frac{d}{ds}\Big|_{s=0} \exp_{LG}(sX) R_{s\alpha} \gamma R_\phi = \frac{d}{ds}\Big|_{s=0} \exp_{LG}(sX) \gamma_{s\alpha} R_{s\alpha+\phi} = X\gamma - \alpha \partial_\theta \gamma + i\alpha d R_\phi \tag{1.4.10}$$

whence $\phi(t) = \alpha t$ and we must solve $\partial_t \gamma = X\gamma - \alpha\partial_\theta\gamma$ with boundary condition $\gamma(\cdot, 0) \equiv 1$. The corresponding homogeneous equation $\partial_t \gamma = -\alpha \partial_\theta \gamma$ is easily solved by setting $\gamma(\theta, t) = \gamma_0(\theta - \alpha t)$ for some $\gamma_0 \in LG$. Varying the constants, we set $\gamma(\theta, t) = \gamma_0(\theta - \alpha t, t)$ and the original equation yields, in terms of γ_0, $\partial_t \gamma_0(\theta, t) = X(\theta + \alpha t)\gamma_0(\theta, t)$ *i.e.* in the notation of theorem 1.4.1, $\dot{\gamma}_0(t) = A(t)\gamma_0(t)$, $\gamma_0(0) = 1$, where $A : \mathbb{R} \to L\mathfrak{g}$ is given by $A(t) = X_{-\alpha t}$. Thus, if we embed G in a space of matrices $M_m(\mathbb{C})$ and LG as a closed subspace of the Fréchet space $C^\infty(S^1, M_m(\mathbb{C})) = \bigcap_k C^k(S^1, M_m(\mathbb{C}))$, then

$$\gamma_0(1) = \prod_{1 \geq \tau \geq 0} \mathrm{Exp}(X_{-\alpha\tau} d\tau) = \lim_{n\to\infty} \exp\Big(\frac{X_{-\alpha}}{n}\Big) \cdots \exp\Big(\frac{X_{-2\alpha/n}}{n}\Big) \exp\Big(\frac{X_{-\alpha/n}}{n}\Big) \tag{1.4.11}$$

where the right-side converges in each Banach space $C^k(S^1, M_m(\mathbb{C}))$ and hence in LG. (1.4.9) now follows from $\gamma(\theta, 1) = \gamma_0(\theta - \alpha, 1)$. The continuity of the exponential map is a direct corollary of (1.4.8) $\diamond$

Corollary 1.4.3. *For any $X \in L\mathfrak{g}$ and $\theta \in \mathbb{R}$, we have*

$$\exp_{LG \rtimes \mathrm{Rot}\, S^1}(X + i\theta d) = \lim_{n\to\infty} \Big(\exp_{LG}\Big(\frac{X}{n}\Big) \exp_{\mathrm{Rot}\, S^1}\Big(\frac{i\theta d}{n}\Big)\Big)^n \tag{1.4.12}$$

Proof. We have

$$\Big(\exp_{LG}\Big(\frac{X}{n}\Big) \exp_{\mathrm{Rot}\, S^1}\Big(\frac{i\theta d}{n}\Big)\Big)^n = \exp_{LG}\Big(\frac{X}{n}\Big) \exp_{LG}\Big(\frac{X_{\theta\cdot 1/n}}{n}\Big) \cdots \exp_{LG}\Big(\frac{X_{\theta\cdot(n-1)/n}}{n}\Big) R_\theta \tag{1.4.13}$$

which, by corollary 1.4.2, tends to $\exp_{LG \rtimes \mathrm{Rot}\, S^1}(X + i\theta d)$ as $n \to \infty$ $\diamond$

Remark. It is fairly easy to prove that the integration of one–parameter groups in $LG \rtimes \mathrm{Rot}\, S^1$ is smoothly well–posed and therefore that the exponential map of this group is smooth. The following result will not be used elsewhere.

Proposition 1.4.4. *The exponential map of $L\mathbb{C}^* \rtimes \mathrm{Rot}\, S^1$ is not locally surjective.*

Proof. Let $u+i\alpha d\in L\mathbb{C}\rtimes i\mathbb{R}=\mathrm{Lie}(L\mathbb{C}^*\rtimes\mathrm{Rot}\,S^1)$. Since $L\mathbb{C}^*$ is abelian, (1.4.9) yields

$$\exp_{L\mathbb{C}^*\rtimes\mathrm{Rot}\,S^1}(u+i\alpha d)=e^{\int_0^1 u_{\alpha\tau}d\tau}R_\alpha \tag{1.4.14}$$

Taking logarithms on each of the factors of $L\mathbb{C}^*\rtimes\mathrm{Rot}\,S^1$, we see that (1.4.14) is locally surjective iff the following map is

$$\mathcal{E}:L\mathbb{C}\rtimes i\mathbb{R}\rightarrow L\mathbb{C}\rtimes i\mathbb{R},\ u+i\alpha d\rightarrow\int_0^1 u(\cdot-\alpha\tau)d\tau+i\alpha d \tag{1.4.15}$$

Write $u=\sum_k a_ke^{ik\theta}$, then, for $\alpha\neq 0$,

$$\mathcal{E}(u+i\alpha d)-i\alpha d=a_0+\sum_{k\neq 0}a_ke^{ik\theta}\frac{1-e^{-ik\alpha}}{ik\alpha} \tag{1.4.16}$$

and it follows that, for any $b\in\mathbb{C}^*$ and $k\in\mathbb{Z}^*$, the elements $be^{ik\theta}+\frac{2\pi}{k}id$, which may be taken arbitrarily close to the origin, are not in the image of $\mathcal{E}$ $\diamond$

1.5. The action of LG on $\mathcal{H}^\infty$.

The invariance of $\mathcal{H}^\infty$ under the action of LG depends upon the following considerations. Let $\gamma\in LG$ and consider the smooth one–parameter group in $LG\rtimes\mathrm{Rot}\,S^1$ given by $\theta\rightarrow\gamma R_\theta\gamma^{-1}=\gamma\gamma_\theta^{-1}R_\theta$. Since $0=(\dot{\overline{\gamma\gamma^{-1}}})=\dot\gamma\gamma^{-1}+\gamma\dot{\gamma}^{-1}$, its derivative at 0 is $id+\dot\gamma\gamma^{-1}$ and therefore, by uniqueness

$$\gamma R_\theta\gamma^{-1}=\exp_{LG\rtimes\mathrm{Rot}\,S^1}(\theta(id+\dot\gamma\gamma^{-1})) \tag{1.5.1}$$

As shown below, this implies that $\pi(\gamma)e^{id\theta}\pi(\gamma)^*=e^{\theta(id+\pi(\dot\gamma\gamma^{-1}))}$ in $PU(\mathcal{H})$ and therefore, up to an additive term, $\pi(\gamma)d\pi(\gamma)^*=d-i\pi(\dot\gamma\gamma^{-1})$. Thus, the perturbed operator $\pi(\gamma)d\pi(\gamma)^*$ is equal to d up to terms which, by proposition 1.2.1 are of lower order with respect to the scale $\mathcal{H}^s$ and it follows from this that $\mathcal{H}^\infty$ is invariant under LG. A detailed argument follows.

Extending our former notation, we denote by $\pi(X)\in\mathrm{End}(\mathcal{H}^\infty)$, $X=Y+i\theta d\in L\mathfrak{g}\rtimes i\mathbb{R}$, the operator $\pi(Y)+i\theta d$ so that we get a projective representation of $L\mathfrak{g}\rtimes i\mathbb{R}$ on $\mathcal{H}^\infty$ satisfying

$$[\pi(X_1),\pi(X_2)]=\pi([X_1,X_2])+i\ell B(X_1,X_2) \tag{1.5.2}$$

where

$$B(Y_1+i\phi_1d,Y_2+i\phi_2d)=B(Y_1,Y_2)=\int_0^{2\pi}\frac{d\theta}{2\pi}\langle Y_1,\dot Y_2\rangle \tag{1.5.3}$$

Extending the Sobolev norms $|\cdot|_t$ to $L\mathfrak{g}\rtimes i\mathbb{R}$ by $|Y+i\theta d|_t=|Y|_t+|\theta|$ and using proposition 1.2.1, we get

$$\|\pi(X)\xi\|_s=\|(1+L_0)^s\pi(X)\xi\|\leq C|Y|_{|s|+\frac12}\|\xi\|_{s+\frac12}+|\theta|\|\xi\|_{s+1}\leq\max(1,C)|X|_{|s|+\frac12}\|\xi\|_{s+1} \tag{1.5.4}$$

so that the $\pi(X)$ extend to bounded operators $\mathcal{H}^s\rightarrow\mathcal{H}^{s-1}$ and are essentially skew–adjoint on $\mathcal{H}^\infty$ by Nelson's commutator theorem [**Ne2**]. The following extends proposition 1.3.2 to the Lie algebra of $LG\rtimes\mathrm{Rot}\,S^1$

Lemma 1.5.1. *For any $X\in L\mathfrak{g}\rtimes i\mathbb{R}d$, the following holds in $PU(\mathcal{H})$*

$$\pi(\exp_{LG\rtimes\mathrm{Rot}\,S^1}X)=e^{\pi(X)} \tag{1.5.5}$$

Proof. Let $X=Y+i\theta d\in L\mathfrak{g}\rtimes i\mathbb{R}$. Then, using Trotter's formula [**Ch**], proposition 1.3.2 and corollary 1.4.3

$$e^{\pi(Y)+i\theta d}=\lim_{n\rightarrow\infty}(e^{\frac{\pi(Y)}{n}}e^{i\frac{\theta}{n}d})^n=\lim_{n\rightarrow\infty}\pi(\exp_{LG}\Big(\frac{Y}{n}\Big)\exp_{\mathrm{Rot}\,S^1}\Big(\frac{i\theta d}{n}\Big))^n=\pi(\exp_{LG\rtimes\mathrm{Rot}\,S^1}(Y+i\theta d)) \tag{1.5.6}$$

$\diamond$

LEMMA 1.5.2. *The following operator identities hold for any $\gamma \in LG$ and $X \in L\mathfrak{g}$*

$$\pi(\gamma)\pi(X)\pi(\gamma)^* = \pi(\mathrm{Ad}(\gamma)X) + ic_1(\gamma, X) \tag{1.5.7}$$

$$\pi(\gamma)d\pi(\gamma)^* = d - i\pi(\dot{\gamma}\gamma^{-1}) + c_2(\gamma) \tag{1.5.8}$$

for some real-valued functions c_1, c_2.

PROOF. Let $\gamma \in LG$ and $X \in L\mathfrak{g} \rtimes i\mathbb{R}$. Then, by lemma 1.5.1, the following holds in $PU(\mathcal{H})$

$$\begin{aligned}\pi(\gamma)e^{t\pi(X)}\pi(\gamma)^* &= \pi(\gamma)\pi(\exp_{LG\rtimes \mathrm{Rot}\, S^1}(tX))\pi(\gamma^{-1}) \\ &= \pi(\gamma \exp_{LG\rtimes \mathrm{Rot}\, S^1}(tX)\gamma^{-1}) \\ &= \pi(\exp_{LG\rtimes \mathrm{Rot}\, S^1}(t\,\mathrm{Ad}(\gamma)X)) \\ &= e^{t\pi(\mathrm{Ad}(\gamma)X)}\end{aligned} \tag{1.5.9}$$

and consequently $\pi(\gamma)e^{t\pi(X)}\pi(\gamma)^* = \lambda(t)e^{t\pi(\mathrm{Ad}(\gamma)X)}$ where λ is a continuous homomorphism $\mathbb{R} \to \mathbb{T}$ so that $\lambda(t) = e^{itc(\gamma,X)}$. The claimed formulae now follow from Stone's theorem $\diamond$

The following result is due to A. Wassermann [**Wa2**] and parallels the corresponding one for positive energy representations of $\mathrm{Diff}(S^1)$ [**GoWa**, thm. 7.4]

PROPOSITION 1.5.3. *The subspaces $\mathcal{H}^n$ are invariant under the action of LG and the corresponding map $LG \times \mathcal{H}^n \to \mathcal{H}^n/\mathbb{T}$ is jointly continuous.*

PROOF. We proceed in several steps.
Invariance of $\mathcal{H}^n$. From lemma 1.5.2 and the fact that $\mathcal{D}(UAU^*) = U\mathcal{D}(A)$ for any operator A and unitary U, we have

$$\begin{aligned}\mathcal{H}^n &= \mathcal{D}((1+d)^n) \\ &\subset \mathcal{D}(\,(1+d-i\pi(\dot{\gamma}\gamma^{-1}) + c_2(\gamma))\big|^n_{\mathcal{D}(1+d)}) \\ &= \pi(\gamma)\mathcal{D}((1+d)^n) \\ &= \pi(\gamma)\mathcal{H}^n\end{aligned} \tag{1.5.10}$$

and therefore $\pi(\gamma^{-1})\mathcal{H}^n \subset \mathcal{H}^n$ for any $\gamma \in LG$.
Continuity of the cocycle c_2. Let $\gamma_m \to \gamma \in LG$ and choose lifts $\pi(\gamma_m)$ and $\pi(\gamma)$ such that $\pi(\gamma_m) \to \pi(\gamma)$ in $U(\mathcal{H})$. Let $\xi \in \mathcal{H}^\infty$, then from lemma 1.5.2,

$$\pi(\gamma_m)(1+d)\xi = (1+d-i\pi(\dot{\gamma}_m\gamma_m^{-1}))\pi(\gamma_m)\xi + c_2(\gamma_m)\pi(\gamma_m)\xi \tag{1.5.11}$$

To deduce from this that $c_2(\gamma_m) \to c_2(\gamma)$, we regularise by multiplying both sides by the compact operator $(1+d)^{-1}$

$$(1+d)^{-1}\pi(\gamma_m)(1+d)\xi = \pi(\gamma_m)\xi - i(1+d)^{-1}\pi(\dot{\gamma}_m\gamma_m^{-1})\pi(\gamma_m)\xi + c_2(\gamma_m)(1+d)^{-1}\pi(\gamma_m)\xi \tag{1.5.12}$$

The first terms on the right and left hand-sides manifestly tend to the corresponding terms for γ. To see that this is the case for the second term on the right hand-side, we use proposition 1.2.1

$$\begin{aligned}&\|(1+d)^{-1}\pi(\dot{\gamma}_m\gamma_m^{-1})\pi(\gamma_m)\xi - (1+d)^{-1}\pi(\dot{\gamma}\gamma^{-1})\pi(\gamma)\xi\| \\ \le&\|(1+d)^{-1}\pi(\dot{\gamma}_m\gamma_m^{-1})(\pi(\gamma_m)-\pi(\gamma))\xi\| \\ +&\|(1+d)^{-1}(\pi(\dot{\gamma}_m\gamma_m^{-1})-\pi(\dot{\gamma}\gamma^{-1}))\pi(\gamma))\xi\| \\ \le&C|\dot{\gamma}_m\gamma_m^{-1}|_{\frac{3}{2}}\|(1+d)^{-\frac{1}{2}}(\pi(\gamma_m)-\pi(\gamma))\xi\| \\ +&C|\dot{\gamma}_m\gamma_m^{-1}-\dot{\gamma}\gamma^{-1}|_{\frac{3}{2}}\|(1+d)^{-\frac{1}{2}}\pi(\gamma))\xi\|\end{aligned} \tag{1.5.13}$$

which tends to zero. Thus, $c_2(\gamma_m)(1+d)^{-1}\pi(\gamma_m)\xi \to c_2(\gamma)(1+d)^{-1}\pi(\gamma)\xi$ and therefore $c_2(\gamma_m) \to c_2(\gamma)$ since $(1+d)^{-1}\pi(\gamma_m)\xi \to (1+d)^{-1}\pi(\gamma)\xi$.

Norm estimates on the $\pi(\gamma)$. Set $A(\gamma) = -i\pi(\gamma^{-1}\dot{\gamma}) + c_2(\gamma^{-1})$ so that, on $\mathcal{H}^\infty$, $\pi(\gamma)^*(1+d)\pi(\gamma) = 1 + d + A(\gamma)$. Notice that, by proposition 1.2.1, $\|A(\gamma)\xi\|_p \leq M_p\|\xi\|_{p+1}$ where $M_p = (C|\gamma^{-1}\dot{\gamma}|_{p+\frac{1}{2}} + |c_2(\gamma^{-1})|)$. If $\xi \in \mathcal{H}^\infty$, we have

$$\|\pi(\gamma)\xi\|_n = \|(1+d)^n\pi(\gamma)\xi\| = \|\pi(\gamma)^*(1+d)^n\pi(\gamma)\xi\| = \|(1+d+A(\gamma))^n\xi\| \tag{1.5.14}$$

and therefore

$$\begin{aligned}
\|\pi(\gamma)\xi\|_n &\leq \sum_{\substack{0\leq p_i+q_i\leq 1\\ p_i+q_i=1}} \|(1+d)^{p_1}A(\gamma)^{q_1}\cdots(1+d)^{p_n}A(\gamma)^{q_n}\xi\| \\
&\leq \sum_{\substack{0\leq p_i+q_i\leq 1\\ p_i+q_i=1}} M_{p_1}^{q_1}\|(1+d)^{1+p_2}A(\gamma)^{q_2}\cdots(1+d)^{p_n}A(\gamma)^{q_n}\xi\| \\
&\leq \sum_{\substack{0\leq p_i+q_i\leq 1\\ p_i+q_i=1}} M_{p_1}^{q_1}M_{1+p_2}^{q_2}\cdots M_{(n-1)+p_n}^{q_n}\|(1+d)^n\xi\| \\
&\leq \sum_{\substack{0\leq p_i+q_i\leq 1\\ p_i+q_i=1}} M_{n-1}^{\sum q_i}\|(1+d)^n\xi\| \\
&= (1+M_{n-1})^n\|\xi\|_n
\end{aligned} \tag{1.5.15}$$

Joint continuity. Fix $\gamma \in LG$. Let $U \subset LG$ be a neighborhood of the identity on which π has a continuous lift to $U(\mathcal{H})$ which we denote by the same letter. Define a continuous lift of π over γU by $\pi(\zeta) = \pi(\gamma)\pi(\gamma^{-1}\zeta)$ where $\pi(\gamma)$ is arbitrary, so that $\pi(\gamma)^*\pi(\zeta) = \pi(\gamma^{-1}\zeta)$. Now for $\xi, \eta \in \mathcal{H}^n$ and $\zeta \in \gamma U$, we have

$$\begin{aligned}
\|\pi(\gamma)\xi - \pi(\zeta)\eta\|_n &\leq \|\pi(\gamma)(\xi-\eta)\|_n + \|(\pi(\gamma)-\pi(\zeta))\eta\|_n \\
&\leq (1+M_{n-1}(\gamma))^n\|\xi-\eta\|_n + (1+M_{n-1}(\gamma))^n\|(1-\pi(\gamma)^*\pi(\zeta))\eta\|_n
\end{aligned} \tag{1.5.16}$$

and

$$\begin{aligned}
\|(1-\pi(\gamma)^*\pi(\zeta))\eta\|_n &= \|(1+d)^n(\pi(\gamma^{-1}\zeta)-1)\eta\| \\
&= \|\pi(\gamma^{-1}\zeta)^*(1+d)^n(\pi(\gamma^{-1}\zeta)-1)\eta\| \\
&\leq \|(1+d+A(\gamma^{-1}\zeta))^n\eta - (1+d)^n\eta\| + \|(1-\pi(\gamma^{-1}\zeta)^*)(1+d)^n\eta\| \\
&\leq ((1+M_{n-1}(\gamma^{-1}\zeta))^n - 1)\|\eta\|_n + \|(1-\pi(\gamma^{-1}\zeta)^*)(1+d)^n\eta\|
\end{aligned} \tag{1.5.17}$$

where the last inequality is proved like (1.5.15). Since c_2 is continuous and $c_2(1) = 0$ the above tends to zero as $\zeta \to \gamma$ and $\eta \to \xi$ $\diamondsuit$

1.6. The crossed homomorphism corresponding to $\mathcal{H}^\infty$.

We compute below the crossed homomorphism corresponding to the joint action of LG and $L\mathfrak{g} \rtimes i\mathbb{R}d$ on $\mathcal{H}^\infty$, namely the map $c : LG \times (L\mathfrak{g} \rtimes i\mathbb{R}d) \to \mathbb{R}$ satisfying

$$\pi(\gamma)\pi(X)\pi(\gamma)^* = \pi(\gamma X\gamma^{-1}) + ic(\gamma, X) \tag{1.6.1}$$

as show that it agrees with the one given by Pressley and Segal [**PS**, 4.9.4]. We shall need two preliminary results.

PROPOSITION 1.6.1. *Let $\mathcal{G}$ be a Fréchet Lie group with smooth exponential map. For any $X, Y \in \mathrm{Lie}\,\mathcal{G}$, the left and right logarithmic derivatives of the exponential map at X are given by*

$$D_X^L \exp_{\mathcal{G}} Y := \left.\frac{d}{ds}\right|_{s=0} \exp_{\mathcal{G}}(-X)\exp_{\mathcal{G}}(X+sY) = \int_0^1 \exp_{\mathcal{G}}(-tX)Y\exp_{\mathcal{G}}(tX)dt \tag{1.6.2}$$

$$D_X^R \exp_{\mathcal{G}} Y := \left.\frac{d}{ds}\right|_{s=0} \exp_{\mathcal{G}}(X+sY)\exp_{\mathcal{G}}(-X) = \int_0^1 \exp_{\mathcal{G}}(tX)Y\exp_{\mathcal{G}}(-tX)dt \tag{1.6.3}$$

PROOF. We use Duhamel's formula [1]. Let $A, B \in \mathrm{Lie}\,\mathcal{G}$ and set $f(t) = \exp_{\mathcal{G}}(-tA)\exp_{\mathcal{G}}(tB)$ so that $\dot{f} = \exp_{\mathcal{G}}(-tA)(B-A)\exp_{\mathcal{G}}(tB)$. If ϕ is a smooth, real-valued function defined on a neighborhood of $f([0,1])$, then

$$\phi(\exp_{\mathcal{G}}(-A)\exp_{\mathcal{G}}(B)) - \phi(1) = \int_0^1 \frac{d}{dt}\phi\circ f\,dt = \int_0^1 \exp_{\mathcal{G}}(-tA)(B-A)\exp_{\mathcal{G}}(tB)\phi\,dt \tag{1.6.4}$$

Set now $A = X$, $B = X + sY$ and let ϕ be defined and smooth near the identity. Then,

$$\begin{aligned} D^L_X \exp_{\mathcal{G}} Y\phi &= \left.\frac{d}{ds}\right|_{s=0} \phi(\exp_{\mathcal{G}}(-X)\exp_{\mathcal{G}}(X+sY)) \\ &= \left.\frac{d}{ds}\right|_{s=0} s\int_0^1 \exp_{\mathcal{G}}(-tX)Y\exp_{\mathcal{G}}(t(X+sY))\phi\,dt \\ &= \int_0^1 \exp_{\mathcal{G}}(-tX)Y\exp_{\mathcal{G}}(tX)\phi\,dt \end{aligned} \tag{1.6.5}$$

and (1.6.2) follows since the above holds for any ϕ smooth near 1. The formula for the right logarithmic derivative follows similarly $\diamond$

COROLLARY 1.6.2. *Let $X \in L\mathfrak{g}$ and $\gamma = \exp_{LG}(X)$. Then, the left and right logarithmic derivatives of γ, seen as a smooth path in G are given by*

$$\gamma^{-1}\dot{\gamma} = \int_0^1 \exp_{LG}(-tX)\dot{X}\exp_{LG}(tX)dt \tag{1.6.6}$$

$$\dot{\gamma}\gamma^{-1} = \int_0^1 \exp_{LG}(tX)\dot{X}\exp_{LG}(-tX)dt \tag{1.6.7}$$

PROOF. Recall that if $p : I \to G$ is a smooth path, the left and right logarithmic derivatives are the paths $I \to \mathfrak{g}$ defined by

$$p^{-1}\dot{p}(t) := \left.\frac{d}{dh}\right|_{h=0} p^{-1}(t)p(t+h) \qquad \text{and} \qquad \dot{p}p^{-1}(t) := \left.\frac{d}{dh}\right|_{h=0} p(t+h)p^{-1}(t) \tag{1.6.8}$$

so that $\dot{p}p^{-1} = \mathrm{Ad}(p)p^{-1}\dot{p}$. Therefore, if $\gamma(\theta) = \exp_G(X(\theta))$, the chain rule and proposition 1.6.1 for $\mathcal{G} = G$ show that the left and right hand-sides of (1.6.6) and (1.6.7) are equal pointwise $\diamond$

THEOREM 1.6.3. *For any $X \in L\mathfrak{g}$ and $\gamma \in LG$, the following identities hold on $\mathcal{H}^\infty$*

$$\pi(\gamma)\pi(X)\pi(\gamma)^* = \pi(\mathrm{Ad}(\gamma)X) - i\ell\int_0^{2\pi} \langle\gamma^{-1}\dot{\gamma}, X\rangle\frac{d\theta}{2\pi} \tag{1.6.9}$$

$$\pi(\gamma)d\pi(\gamma)^* = d - i\pi(\dot{\gamma}\gamma^{-1}) - \frac{\ell}{2}\int_0^{2\pi} \langle\gamma^{-1}\dot{\gamma}, \gamma^{-1}\dot{\gamma}\rangle\frac{d\theta}{2\pi} \tag{1.6.10}$$

PROOF. Specialising lemma 1.5.2 to $\mathcal{H}^\infty$, we find for any $\gamma \in LG$ and $X \in L\mathfrak{g} \rtimes i\mathbb{R}d$

$$\pi(\gamma)\pi(X)\pi(\gamma)^* = \pi(\mathrm{Ad}(\gamma)X) + ic(\gamma, X) \tag{1.6.11}$$

from which (1.6.9) follows by the same proof as that of (iii) of proposition I.3.2.2. (1.6.10) is more difficult to establish because d does not lie in the derived subalgebra of $L\mathfrak{g} \rtimes i\mathbb{R}d$. To compute the values $c(\gamma, id)$, we shall show that the restriction of c to one–parameter subgroups in LG satisfies a simple linear, first order ODE which may be solved explicitly. It then easily follows that the right hand-sides of (1.6.9)–(1.6.10) and c coincide on the image of $\exp_{LG}$ and hence on LG because both are crossed homomorphisms $LG \to (L\mathfrak{g} \rtimes i\mathbb{R}d)^*$ and $\exp_{LG}(L\mathfrak{g})$ generates LG. We proceed in several steps.

Continuity of c. We claim that c is jointly continuous in its arguments or, equivalently that it can be viewed as a continuous map $c : LG \to (L\mathfrak{g} \rtimes i\mathbb{R}d)^*$ where the latter is the locally convex space of all continuous, linear forms on $L\mathfrak{g} \rtimes i\mathbb{R}d$ endowed with the weak-* topology. *i.e.* the topology of pointwise convergence. To see this, let $(\gamma, X) \in LG \times (L\mathfrak{g} \rtimes i\mathbb{R}d)$ and choose a neighborhood $\gamma \in U$ such that

[1] This idea is due to A. Selby

$\pi : U \to PU(\mathcal{H})$ possesses a continuous lift to $U(\mathcal{H})$. Fix $\xi \in \mathcal{H}^\infty$ and for $\zeta \in U$ and $Y \in L\mathfrak{g} \rtimes i\mathbb{R}d$, write

$$ic(\zeta, Y)\pi(\zeta)\xi = \pi(\zeta)\pi(Y)\xi - \pi(\mathrm{Ad}(\zeta)Y)\pi(\zeta)\xi \tag{1.6.12}$$

so that

$$\begin{aligned}|c(\zeta, Y) - c(\gamma, X)| &\leq \|c(\zeta, Y)\pi(\zeta)\xi - c(\gamma, X)\pi(\gamma)\xi\| + |c(\gamma, X)|\|\pi(\gamma)\xi - \pi(\zeta)\xi\| \\ &\leq \|\pi(\zeta)\pi(Y)\xi - \pi(\gamma)\pi(X)\xi\| + \|\pi(\mathrm{Ad}(\zeta)Y)\pi(\zeta)\xi - \pi(\mathrm{Ad}(\gamma)X)\pi(\gamma)\xi\| \\ &+ |c(\gamma, X)|\|\pi(\gamma)\xi - \pi(\zeta)\xi\| \\ &\leq \|\pi(\zeta)(\pi(Y) - \pi(X))\xi\| + \|(\pi(\zeta) - \pi(\gamma))\pi(X)\xi\| \\ &+ \|\pi(\mathrm{Ad}(\zeta)Y)(\pi(\zeta) - \pi(\gamma))\xi\| + \|(\pi(\mathrm{Ad}(\zeta)Y) - \pi(\mathrm{Ad}(\gamma)X))\pi(\gamma)\xi\| \\ &+ |c(\gamma, X)|\|\pi(\gamma)\xi - \pi(\zeta)\xi\| \\ &\leq C|Y - X|_{\frac{1}{2}}\|(1 + d)\xi\| + \|(\pi(\zeta) - \pi(\gamma))\pi(X)\xi\| \\ &+ C|\mathrm{Ad}(\zeta)Y|_{\frac{1}{2}}\|(1 + d)(\pi(\zeta) - \pi(\gamma))\xi\| + C|\mathrm{Ad}(\zeta)Y - \mathrm{Ad}(\gamma)X|_{\frac{1}{2}}\|(1 + d)\pi(\gamma)\xi\| \\ &+ |c(\gamma, X)|\|\pi(\gamma)\xi - \pi(\zeta)\xi\|\end{aligned} \tag{1.6.13}$$

which, in view of the continuity of the action of LG on $\mathcal{H}^1$, can be made arbitrarily small.

Functional Equation. The following is an immediate corollary of (1.6.11)

$$c(\gamma_2\gamma_1, X) = c(\gamma_1, X) + c(\gamma_2, \mathrm{Ad}(\gamma_1)X) \tag{1.6.14}$$

in other words, c is a continuous crossed homomorphism $LG \to (L\mathfrak{g} \rtimes i\mathbb{R}d)^*$.

Differentiability at 1. We claim that the restriction of c to one-parameter subgroups $\gamma_t = \exp_{LG}(tY)$ is differentiable at $t = 0$. To see this, notice that by proposition 1.5.1, the unitary $e^{t\pi(Y)}$ is a lift of $\pi(\gamma_t)$. Applying both sides of (1.6.11) to $\xi \in \mathcal{H}^\infty$ and developing in Taylor series, we get

$$\pi(\mathrm{Ad}(\gamma_t)X)\xi = \pi(X + t[Y, X] + o(t))\xi = \pi(X) + t\pi([Y, X])\xi + o(t) \tag{1.6.15}$$

where $\pi(o(t))\xi = o(t)$ follows from $\|\pi(Z)\xi\| \leq C|Z|_{\frac{1}{2}}\|(1 + d)\xi\|$. Next

$$\begin{aligned}e^{t\pi(Y)}\pi(X)e^{-t\pi(Y)}\xi &= e^{t\pi(Y)}\pi(X)(\xi - t\pi(Y)\xi + R(t)) \\ &= e^{t\pi(Y)}(\pi(X)\xi - t\pi(X)\pi(Y)\xi + o(t)) \\ &= \pi(X)\xi + t[\pi(Y), \pi(X)]\xi + o(t)\end{aligned} \tag{1.6.16}$$

since $\pi(X)R(t) = o(t)$. Indeed, the remainder term $R(t)$ is equal to $t^2 \int_0^1 du u \int_0^1 dv \pi(Y)^2 e^{-tuv\pi(Y)}\xi$ and therefore

$$\|\pi(X)R(t)\| \leq Ct^2|X|_{\frac{1}{2}}\|(1 + d) \int_0^1 du u \int_0^1 dv \pi(Y)^2 e^{-tuv\pi(Y)}\xi\| \tag{1.6.17}$$

which is an $o(t)$ since the unitaries $e^{-tuv\pi(Y)}$ are uniformly bounded in the $\|\cdot\|_2$ norm by proposition 1.5.3. Summarising,

$$ic(\gamma_t, X)\xi = t([\pi(Y), \pi(X)] - \pi([X, Y]))\xi + o(t) = it\ell B(X, Y)\xi + o(t) \tag{1.6.18}$$

Thus, $c(\gamma_t)$ is differentiable at 0 and

$$\partial_t|_{t=0}\, c(\gamma(t), X) = \ell B(Y, X) \tag{1.6.19}$$

Ordinary Differential Equation. We shall now show that $c(\gamma_t)$ is differentiable at any $t \in \mathbb{R}$ and that

$$\partial_t c(\gamma_t, X) = \ell B(Y, X) + c(\gamma_t, [Y, X]) \tag{1.6.20}$$

Using (1.6.14), we get $c(\gamma_{t+h}, X) = c(\gamma_h, X) + c(\gamma_t, \mathrm{Ad}(\gamma_h)X)$ and therefore

$$\frac{1}{h}\Big(c(\gamma_{t+h}, X) - c(\gamma_t, X)) = \frac{1}{h}c(\gamma_h, X) + c(\gamma_t, \frac{1}{h}(\mathrm{Ad}(\gamma_h)X - X)) \tag{1.6.21}$$

Letting $h \to 0$ and using (1.6.19) and continuity in the second variable yields (1.6.20).

Formal solution of the ODE. Somewhat changing notation, we write (1.6.20) as the inhomogeneous ODE

$$\dot{c}_t = i_Y B - Y c_t \tag{1.6.22}$$

where $i_Y B(X) = B(Y, X)$ and Yc_t denotes the coadjoint action of LG on $(L\mathfrak{g} \rtimes i\mathbb{R}d)^*$. The underlying homogeneous equation is formally solved by $c_t = e^{-tY} c_0$. Varying the constants, we set $c_t = e^{-tY} c_0(t)$, so that (1.6.22) reads, in terms of c_0, $e^{-tY} \dot{c}_0 = i_Y B$, $c_0(0) = 0$ whence $c_0(t) = \int_0^t e^{\tau Y} i_Y B \, d\tau$. In other words, reverting to our former notation

$$c(\gamma_t, X) = \ell B(Y, \int_0^t \mathrm{Ad}(\gamma_{t-\tau}) X \, d\tau) = \ell B(Y, \int_0^t \mathrm{Ad}(\gamma_\tau) X \, d\tau) \tag{1.6.23}$$

Existence and uniqueness of solution. Using $\frac{d}{dt} \mathrm{Ad}(\gamma_t) X = [Y, \mathrm{Ad}(\gamma_t) X]$, it is easy to verify that (1.6.23) defines a $C^1(\mathbb{R}, (Lg \rtimes i\mathbb{R}d)^*)$ solution of (1.6.20) with initial condition $c_0 = 0$. If c' is another solution, $\kappa_t = c_t - c'_t$ satisfies $\dot{\kappa}_t(X) = \kappa_t([Y, X])$, $\kappa_0 = 0$. Setting $K_t(X) = \kappa_t(\mathrm{Ad}(\gamma_{-t})X)$, we see that $\dot{K}_t = 0$ whence K is locally constant since it takes values in a locally convex vector space. Thus, $\kappa \equiv 0$ and $c = c'$.

Reinterpretation of the solution. We derive now a formula for c which naturally extends to the whole of LG. Let $Y, X \in L\mathfrak{g}$ and set $\gamma_t = \exp_{LG}(tY)$. By (1.6.23) we have, in view of the Ad_G invariance of $\langle \cdot, \cdot \rangle$ and corollary 1.6.2

$$\begin{aligned} c(\exp_{LG} Y, X) &= -\ell \int_0^{2\pi} \int_0^1 dt \langle \dot{Y}, \gamma_t X \gamma_{-t} \rangle \frac{d\theta}{2\pi} \\ &= -\ell \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^1 dt \langle \gamma_{-t} \dot{Y} \gamma_t, X \rangle \frac{d\theta}{2\pi} \\ &= -\ell \int_0^{2\pi} \frac{d\theta}{2\pi} \langle \gamma_1^{-1} \dot{\gamma}_1, X \rangle \frac{d\theta}{2\pi} \end{aligned} \tag{1.6.24}$$

Similarly, using $\mathrm{Ad}(\gamma) id = id + \dot{\gamma}\gamma^{-1}$, corollary 1.6.2 and the symmetry of $\langle \cdot, \cdot \rangle$,

$$\begin{aligned} c(\exp_{LG} Y, id) &= -\ell \int_0^{2\pi} \int_0^1 dt \langle \dot{Y}, \dot{\gamma}_t \gamma_t^{-1} \rangle \frac{d\theta}{2\pi} \\ &= -\ell \int_0^{2\pi} \int_0^1 \int_0^t \langle \dot{Y}, \gamma_u \dot{Y} \gamma_{-u} \rangle du dt \frac{d\theta}{2\pi} \\ &= -\frac{\ell}{2} \int_0^{2\pi} \left(\int_0^1 \int_0^t \langle \dot{Y}, \gamma_u \dot{Y} \gamma_{-u} \rangle du dt + \int_0^1 \int_{-t}^0 \langle \dot{Y}, \gamma_u \dot{Y} \gamma_{-u} \rangle du dt \right) \frac{d\theta}{2\pi} \\ &= -\frac{\ell}{2} \int_0^{2\pi} \left(\iint_{0 \le v \le t \le 1} \langle \dot{Y}, \gamma_{t-v} \dot{Y} \gamma_{v-t} \rangle dv dt + \iint_{0 \le t \le v \le 1} \langle \dot{Y}, \gamma_{t-v} \dot{Y} \gamma_{v-t} \rangle dv dt \right) \frac{d\theta}{2\pi} \\ &= -\frac{\ell}{2} \int_0^{2\pi} \langle \gamma_1^{-1} \dot{\gamma}_1, \gamma_1^{-1} \dot{\gamma}_1 \rangle \frac{d\theta}{2\pi} \end{aligned} \tag{1.6.25}$$

Therefore, for any γ in the image of $\exp_{LG}$, we have

$$c(\gamma, X + i\phi d) = -\ell \int_0^{2\pi} \langle \gamma^{-1} \dot{\gamma}, X \rangle \frac{d\theta}{2\pi} - \frac{\ell \phi}{2} \int_0^{2\pi} \langle \gamma^{-1} \dot{\gamma}, \gamma^{-1} \dot{\gamma} \rangle \frac{d\theta}{2\pi} \tag{1.6.26}$$

The right hand-side extends in an obvious way to a function $\widetilde{c} : LG \to (L\mathfrak{g} \rtimes i\mathbb{R}d)^*$ which is easily seen to satisfy (1.6.14). Since any solution of that functional equation is uniquely determined by its restriction to a generating set in LG and the image of $\exp_{LG}$ is such a set, we conclude that $c \equiv \widetilde{c}$ ◇

2. Central extensions of LG arising from positive energy representations

We show below that positive energy representations of equal level induce canonically isomorphic central extensions of LG. As explained in chapter I, this basic fact is needed to define the direct sum of these representations and ultimately to show that the category of all positive energy representations at a given level is abelian. Our method shows in fact that these topological central extensions possess the structure of real analytic Fréchet Lie groups, a fact which, surprisingly is easier to establish than their smoothness. It relies on proving that their local multiplication is analytic, which in turn is derived from Nelson's analytic domination theorem [**Ne3**] and the Sobolev estimates of proposition 1.2.1.

Once their analyticity is established, the classification of the central extensions reduces, by an elegant argument of Pressley and Segal [**PS**, 4.4.1] to that of their Lie algebra cocycle which we compute explicitly. As expected, the latter differs from the fundamental cocycle by a factor equal to the level of the representation. An alternative method which proves the smoothness of the central extensions only but also applies to $\mathrm{Diff}(S^1)$ may be found in [**TL1**].

2.1. The infinite dimensional projective unitary group.

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space. We endow the unitary group $U(\mathcal{H})$ with the strong operator topology so that it becomes a complete, metrisable topological group which moreover is contractible [**DiDo**, Lemme 3, page 251]. When the projective unitary group $PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}$ is endowed with the quotient topology, the short exact sequence

$$1 \to \mathbb{T} \to U(\mathcal{H}) \xrightarrow{p} PU(\mathcal{H}) \to 1 \tag{2.1.1}$$

possesses local continuous sections (see below) and may therefore be regarded as a continuous, principal $\mathbb{T}$-bundle over $PU(\mathcal{H})$. The associated long exact sequence of homotopy groups shows that $PU(\mathcal{H})$ is connected and simply–connected and therefore that (2.1.1) does not possess local homomorphic sections, contrary to its finite–dimensional analogues. Indeed, any such would extend, by the local monodromy principle, to a global section yielding an isomorphism $U(\mathcal{H}) \cong PU(\mathcal{H}) \times \mathbb{T}$ in contradiction with the contractibility of $U(\mathcal{H})$.

Local continuous sections of (2.1.1) may be exhibited as follows [**Ba**]. Fix $0 \neq \xi \in \mathcal{H}$ and consider the open set

$$U_\xi = \{[u] \in PU(\mathcal{H}) |\, |(u\xi, \xi)| > 0\} \tag{2.1.2}$$

where $[u]$ is the equivalence class of $u \in U(\mathcal{H})$ in $PU(\mathcal{H})$, so that $p^{-1}(U_\xi) = \{u \in U(\mathcal{H}) |\, (u\xi, \xi) \neq 0\}$. Define a function

$$\alpha_\xi : p^{-1}(U_\xi) \longrightarrow \mathbb{T}, \quad u \longrightarrow \alpha_\xi(u) = \frac{(u\xi, \xi)}{|(u\xi, \xi)|} \tag{2.1.3}$$

and notice that $\alpha_\xi(e^{i\theta}u) = e^{i\theta}\alpha_\xi(u)$ so that the map $\phi : p^{-1}(U_\xi) \to U_\xi \times \mathbb{T}$, $\phi(u) = ([u], \alpha_\xi(u))$ is a $\mathbb{T}$-equivariant local trivialisation with inverse $\phi^{-1}([u], z) = u\alpha_\xi(u)^{-1}z$. If $1 \in V_\xi \subset U_\xi$ is a neighborhood such that $V_\xi \cdot V_\xi \subset U_\xi$ and $V_\xi^{-1} = V_\xi$, we may use ϕ to define a local multiplication and inversion on $V_\xi \times \mathbb{T}$ by

$$x \star y = \phi(\phi^{-1}x \cdot \phi^{-1}y) \tag{2.1.4}$$

$$\mathcal{I}(x) = \phi\big((\phi^{-1}x)^{-1}\big) \tag{2.1.5}$$

where $\cdot$ is the multiplication in $U(\mathcal{H})$. Explicitly, using $\alpha_\xi(u^*) = \overline{\alpha_\xi(u)}$,

$$([u], z) \star ([v], w) = \Big([u \cdot v], zw\frac{\alpha_\xi(uv)}{\alpha_\xi(u)\alpha_\xi(v)}\Big) \tag{2.1.6}$$

$$\mathcal{I}([u], z) = ([u^{-1}], z^{-1}) \tag{2.1.7}$$

where the quotient $\alpha_\xi(uv)\alpha_\xi(u)^{-1}\alpha_\xi(v)^{-1}$ is independent of the choice of the lifts u, v of $[u], [v] \in PU(\mathcal{H})$.

2.2. The central extensions $\mathcal{L}G$ and their local multiplication.

Let $\pi : LG \to PU(\mathcal{H})$ be a positive energy representation of LG and $\mathcal{L}G = \pi^*U(\mathcal{H})$ the topological central extension

$$1 \to \mathbb{T} \to \mathcal{L}G \xrightarrow{p} LG \to 1 \tag{2.2.1}$$

obtained by pulling back (2.1.1) by π. Explicitly,

$$\mathcal{L}G = \{(g, u) \in LG \times U(\mathcal{H}) |\, \pi(g) = [u]\} \tag{2.2.2}$$

As in §2.1, we may trivialise $\mathcal{L}G$ over $\pi^{-1}(U_\xi)$ to obtain a homeomorphism $p^{-1}\pi^{-1}(U_\xi) \rightarrow \pi^{-1}(U_\xi)\times\mathbb{T}$ with inverse $(\gamma, z) \rightarrow (\gamma, \pi(\gamma)\alpha_\xi(\pi(\gamma))^{-1}z)$ and corresponding local multiplication and inversion

$$(\zeta, z) \star (\gamma, w) = \Big(\zeta\gamma, zw\frac{\alpha_\xi(\pi(\zeta)\pi(\gamma))}{\alpha_\xi(\pi(\zeta))\alpha_\xi(\pi(\gamma))}\Big) \tag{2.2.3}$$

$$\mathcal{I}(\zeta, z) = (\zeta^{-1}, z^{-1}) \tag{2.2.4}$$

We will prove in §2.4 that ξ may be chosen so that $\star$ is (real) analytic near the identity. This in turn shows that the above local trivialisation of $\mathcal{L}G$ extends to an analytic atlas and therefore that (2.2.1) is a real analytic central extension of LG in view of the following elementary

LEMMA 2.2.1. *Let $\mathcal{G}$ be a connected topological group homeomorphic near 1 to an open subset of a Fréchet space E. If the transported local multiplication $\star : E\times E \rightarrow E$ and inversion $\mathcal{I} : E \rightarrow E$ are real analytic maps, $\mathcal{G}$ possesses the structure of a real analytic Fréchet Lie group modelled on E.*

PROOF. Let $\phi : U \subset \mathcal{G} \rightarrow \phi(U) \subset E$ be the local homeomorphism and $1 \in V \subset U$ a neighborhood such that $V\cdot V \subset U$ and $V^{-1} = V$. Assume for simplicity that $\phi(1) = 0$. The transported multiplication $\star : \phi(V)\times\phi(V) \rightarrow \phi(U)$ and inversion $\mathcal{I} : \phi(V) \rightarrow \phi(V)$ are defined by

$$x \star y = \phi(\phi^{-1}x\cdot\phi^{-1}y) \tag{2.2.5}$$

$$\mathcal{I}(x) = \phi\big((\phi^{-1}x)^{-1}\big) \tag{2.2.6}$$

and are, by assumption, analytic. Consider a neighborhood $1 \in W \subset V$ such that $W\cdot W \subset V$ and $W^{-1} = W$. We define an atlas $\mathcal{A} = \Big\{(W_g, \phi_g)\Big\}$ indexed by $g \in \mathcal{G}$ by setting $W_g = gW$, $\phi_g : W_g \rightarrow \phi(W)$, $\phi_g(h) = \phi(g^{-1}h)$ so that $\phi_g^{-1}(x) = g\phi^{-1}(x)$. We claim that $(\mathcal{G}, \mathcal{A})$ has the structure of an analytic Fréchet Lie group.
Analyticity of the atlas. Assume $W_{g_1}\cap W_{g_2} \neq 0$. Then, $g_2^{-1}g_1 \in W\cdot W^{-1} \subset V$ may be written as $\phi^{-1}(y)$ with $y \in \phi(V)$ and therefore for $x \in E$ close to the origin, the transition map

$$\tau_{g_2g_1}(x) = \phi_{g_2}\circ\phi_{g_1}^{-1}(x) = \phi(g_2^{-1}g_1\phi^{-1}(x)) = \phi(\phi^{-1}(y)\phi^{-1}(x)) = y \star x \tag{2.2.7}$$

is analytic.
Analyticity of the adjunction. Let $g \in \mathcal{G}$ and define the transported adjunction $\mathcal{A}d(g)$ about $0 \in E$ by $\mathcal{A}d(g)x = \phi(g\phi^{-1}(x)g^{-1})$. If g lies in a suitably small neighborhood $1 \in S \subset W$, $\mathcal{A}d(g)$ is smooth (analytic) near 0 since $\mathcal{A}d(g)x = y \star x \star \mathcal{I}(y)$ with $y = \phi(g)$. Similarly, if $g = g_1\cdots g_k \in S^k$, then $\mathcal{A}d(g) = \mathcal{A}d(g_1)\circ\cdots\circ\mathcal{A}d(g_k)$ is analytic near 1. Since $\mathcal{G}$ is connected, $\bigcup_k S^k = \mathcal{G}$ and $\mathcal{A}d(g)$ is analytic near 0 for any $g \in \mathcal{G}$.
Analyticity of multiplication. Let $g, h \in \mathcal{G}$ and $x, y \in E$ small enough, then

$$\phi_{gh}\Big(\phi_g^{-1}(x)\phi_h^{-1}(y)\Big) = \phi(h^{-1}\phi^{-1}(x)h\phi^{-1}(y)) = (\mathcal{A}d(h^{-1})x)\star y \tag{2.2.8}$$

is analytic.
Analyticity of inversion. Let $g \in \mathcal{G}$ and $x \in E$ small enough, then

$$\phi_{g^{-1}}(\phi_g^{-1}(x)^{-1}) = \phi(g\phi^{-1}(x)^{-1}g^{-1}) = \mathcal{A}d(g)\circ\mathcal{I}(x) \tag{2.2.9}$$

is analytic $\diamondsuit$

2.3. Analytic Domination of $L\mathfrak{g}$ in Positive Energy Representations.

We will need Goodman's refinement of Nelson's fundamental analytic domination theorem [**Ne3**, Thm. 1]

THEOREM 2.3.1. *Let $\{X_i\}_{i\in I}$ be a family of endomorphisms of a normed vector space V and $A \in \operatorname{End}(V)$ such that, for any $\xi \in V$*

$$\|X_i\xi\| \leq \|A\xi\| \tag{2.3.1}$$

$$\|\operatorname{ad}X_{i_n}\cdots\operatorname{ad}X_{i_1}(A)\xi\| \leq n!\|A\xi\| \tag{2.3.2}$$

for any $i \in I$, $n \in \mathbb{N}$ and finite subset $\{i_1, \ldots, i_n\} \subset I$. Then there exists a constant $\widetilde{M} > 0$ such that the inequalities

$$\|X_{i_n} \cdots X_{i_1}\xi\| \leq n!\widetilde{M}^n \tag{2.3.3}$$

hold whenever $\xi \in V$ satisfies $\|A^m\xi\| \leq m!$ for any $m \in \mathbb{N}$.

PROOF. The above is simply a reformulation of lemma 2′ of [**Go**] (where the author simply omitted to state explicitly the independence of $\widetilde{M}$ on ξ). The proof is obtained by combining the sketch proof of lemma 2′ and the proof of lemma 2 in [**Go**] $\diamond$

COROLLARY 2.3.2. *Let $\mathcal{H}$ be a positive energy representation of LG with finite energy subspace $\mathcal{H}^{\text{fin}}$. Then, there exists a constant $\infty > M > 0$ such that for any $\xi \in \mathcal{H}^{\text{fin}}$ and $X_1, \ldots, X_n \in L\mathfrak{g}$, the following inequalities hold*

$$\|\pi(X_1) \cdots \pi(X_n)\xi\| \leq \lambda_\xi |X_1|_{\frac{3}{2}} \cdots |X_n|_{\frac{3}{2}} M^n n! \tag{2.3.4}$$

for some constant $0 < \lambda_\xi < \infty$ depending on ξ.

PROOF. Set $V = \mathcal{H}^\infty$ and $A = 1+d$ in theorem 2.3.1 where d is the infinitesimal generator of rotations on $\mathcal{H}$. We will show that the inequalities (2.3.1)–(2.3.2) hold for a suitable neighborhood of 0 in $L\mathfrak{g}$. From the estimates of proposition 1.2.1 and the commutation relations (1.3.1)–(1.3.2), we find that $\|\pi(X)\xi\| \leq C|X|_{\frac{1}{2}}\|A\xi\|$ for any $\xi \in V$ and, for $n \geq 2$

$$\begin{aligned}
\|\operatorname{ad}\pi(X_n) \cdots \operatorname{ad}\pi(X_1)A\xi\| &\leq C|\operatorname{ad}X_n \cdots \operatorname{ad}X_2\dot{X}_1|_{\frac{1}{2}}\|A\xi\| + \ell|B(X_n, \operatorname{ad}X_{n-1} \cdots \operatorname{ad}X_2\dot{X}_1)|\,\|\xi\| \\
&\leq CC_{\mathfrak{g}}^{n-1}|X_n|_{\frac{1}{2}} \cdots |X_2|_{\frac{1}{2}}|\dot{X}_1|_{\frac{1}{2}}\|A\xi\| + \ell C_{\mathfrak{g}}^{n-2}|X_n|_1|X_{n-1}|_0 \cdots |X_2|_0|\dot{X}_1|_0\|\xi\| \\
&\leq C_g^{n-2}(CC_{\mathfrak{g}} + \ell)|X_n|_{\frac{3}{2}} \cdots |X_1|_{\frac{3}{2}}\|A\xi\|
\end{aligned} \tag{2.3.5}$$

where ℓ is the level of $\mathcal{H}$ and we have used $|\dot{X}|_t \leq |X|_{t+1}$, $|[X,Y]|_t \leq C_{\mathfrak{g}}|X|_t|Y|_t$ and $|B(X,Y)| \leq |X|_1|Y|_0$. Thus, for $r > 0$ small enough, the family $\{\pi(X)|\, X \in L\mathfrak{g}, |X|_{\frac{3}{2}} \leq r\}$ satisfies the inequalities (2.3.1)–(2.3.2) of theorem 2.3.1 for any $\xi \in V$. Let now $\mathcal{H}^{\text{fin}} \ni \xi = \sum_k \xi_k$ be a finite sum of eigenvectors of A with eigenvalues μ_k. Set $\sigma = \max_k \mu_k$ and $\lambda_\xi = \sup_m \frac{\sigma^m}{m!}$, then $\|A^m\lambda_\xi^{-1}\xi\| \leq m!$ for all $m \in \mathbb{N}$ and therefore, for any $X_1, \ldots X_n \in L_{\mathfrak{g}}$

$$\|\pi(X_n) \cdots \pi(X_1)\xi\| = \lambda_\xi r^{-n}|X_n|_{\frac{3}{2}} \cdots |X_1|_{\frac{3}{2}}\|r\frac{\pi(X_n)}{|X_n|_{\frac{3}{2}}} \cdots r\frac{\pi(X_1)}{|X_1|_{\frac{3}{2}}}\frac{\xi}{\lambda_\xi}\| \leq \lambda_\xi|X_n|_{\frac{3}{2}} \cdots |X_1|_{\frac{3}{2}}M^n n! \tag{2.3.6}$$

where $M = \widetilde{M}r^{-1}$ $\diamond$

2.4. $\mathcal{L}G$ as a real analytic Fréchet Lie group.

The following is a simple consequence of the spectral theorem

LEMMA 2.4.1. *Let X, Y be two essentially skew–adjoint operators on a Hilbert space $\mathcal{H}$ defined on an invariant domain $\mathcal{D}$. Denote by $\overline{X}$, $\overline{Y}$ the closures of X and Y and let the unitaries $e^{\overline{X}}$, $e^{\overline{Y}}$ be defined by the spectral theorem. If $\xi \in \mathcal{D}$, the identities*

(i) $\displaystyle\sum_{n\geq 0}\frac{Y^n}{n!}\xi = e^{\overline{Y}}\xi$

(ii) $\displaystyle\sum_{m,n\geq 0}\frac{X^mY^n}{m!n!}\xi = e^{\overline{X}}e^{\overline{Y}}\xi$

hold whenever their left-hand sides are absolutely convergent.

PROOF. We may suppose that X and Y are closed with invariant subdomain $\mathcal{D}$.
(i) By the spectral theorem, we may assume $\mathcal{H} = L^2(\Omega, \mu)$ with Y acting as multiplication by a measurable, $i\mathbb{R}$-valued, μ-almost everywhere finite function f. Then, $\mathcal{D}(Y^n) = \{g \in \mathcal{H}|\, \int_\Omega |f|^{2n}|g|^2 d\mu < \infty\}$.

If $\xi \in \bigcap_n \mathcal{D}(Y^n)$ is such that the left-hand side of (i) is absolutely convergent, the sequence of functions $\eta_N = \sum_{n=0}^N \frac{1}{n!} f^n \xi$ converges to some $\eta \in \mathcal{H}$. Therefore [**Ru**, Theorem 3.12], for almost every $\omega \in \Omega$, $|f(\omega)|, |\xi(\omega)| < \infty$ and $\eta_N(\omega) \to \eta(\omega)$. However,

$$\eta_N(\omega) = \sum_{n=0}^N \frac{1}{n!} f^n(w)\xi(w) \to e^{f(\omega)}\xi(\omega) \tag{2.4.1}$$

whence $\eta = e^f \xi = e^Y \xi$.
(ii) Assume that $\xi \in \mathcal{D}$ is such that the left hand-side of (ii) is absolutely convergent and set $\eta = e^Y \xi$. We will show inductively that $\eta \in \mathcal{D}(X^m)$ and that $X^m \eta = \sum_{n \geq 0} X^m \frac{Y^n}{n!} \xi$ for any $m \in \mathbb{N}$. The case $m = 0$ is settled by (i). If $m \geq 1$, set $\theta_N = \sum_{n=0}^N X^{m-1} \frac{Y^n}{n!} \xi$, then by assumption $\theta_N \in \mathcal{D} \subset \mathcal{D}(X)$ and by induction $\theta_N \to X^{m-1}\eta$. Moreover, by assumption $X\theta_N$ is Cauchy and hence by closedness of X, $X^{m-1}\eta \in \mathcal{D}(X)$, *i.e.* $\eta \in \mathcal{D}(X^m)$ and $X^m \eta = \sum_{n \geq 0} X^m \frac{Y^n}{n!} \xi$. Therefore, $\eta \in \bigcap_m D(X^m)$ and the series $\sum_{m \geq 0} \frac{X^m}{m!} \eta$ is absolutely convergent and therefore equal to $e^X \eta$. Finally,

$$\sum_{m,n \geq 0} \frac{X^m Y^n}{m! n!} \xi = \sum_{m \geq 0} \sum_{n \geq 0} \frac{X^m Y^n}{m! n!} \xi = \sum_{m \geq 0} \frac{X^m}{m!} e^Y \xi = e^X e^Y \xi \tag{2.4.2}$$

$\diamond$

THEOREM 2.4.2. *Let $(\pi, \mathcal{H})$ be a positive energy representation of LG and*

$$1 \to \mathbb{T} \to \mathcal{L}G \to LG \to 1 \tag{2.4.3}$$

the corresponding topological central extension induced by π. Then

(i) *$\mathcal{L}G$ possesses the structure of a real analytic Fréchet Lie group modelled locally on $L\mathfrak{g} \oplus i\mathbb{R}$ such that* (2.4.3) *is a real analytic central extension of LG.*
(ii) *$\mathcal{H}$ possesses a dense set of analytic vectors for the action of LG.*

PROOF. (i) By lemma 2.2.1, we need only prove that the vector ξ used in the local trivialisation of $\mathcal{L}G$ may be chosen so that the local multiplication (2.2.3) is analytic. Fix $\xi \in \mathcal{H}^{\text{fin}}$ and $\delta \leq (2M)^{-1}$ and let $X, Y \in L\mathfrak{g}$, $|X|_{\frac{3}{2}}, |Y|_{\frac{3}{2}} < \delta$. Then, by corollary 2.3.2,

$$\sum_{m,n \geq 0} \frac{\|\pi(X)^m \pi(Y)^n \xi\|}{m! n!} \leq \lambda_\xi \sum_{m,n \geq 0} \binom{m}{n} |X|_{\frac{3}{2}}^m |Y|_{\frac{3}{2}}^n M^{m+n} = \lambda_\xi \sum_{k \geq 0} (M|X|_{\frac{3}{2}} + M|Y|_{\frac{3}{2}})^k < \infty \tag{2.4.4}$$

Therefore, if $B_\delta = \{X \in L\mathfrak{g} | \, |X|_{\frac{3}{2}} < \delta\}$ then, in view of lemma 2.4.1, the function $f_\xi : L\mathfrak{g} \times L\mathfrak{g} \to \mathcal{H}$, $(X, Y) \to e^{\pi(X)} e^{\pi(Y)} \xi$ is analytic on $B_\delta \times B_\delta$. Moreover, since

$$\|e^{\pi(X)} e^{\pi(Y)} \xi - \xi\| \leq \lambda_\xi \sum_{k \geq 1} (M|X|_{\frac{3}{2}} + M|Y|_{\frac{3}{2}})^k \tag{2.4.5}$$

we may take δ small enough so that $(e^{\pi(X)} e^{\pi(Y)} \xi, \xi) \neq 0$ for any $(X, Y) \in B_\delta \times B_\delta$ and it follows that the function

$$(X, Y) \to \frac{\alpha_\xi(e^{\pi(X)} e^{\pi(Y)})}{\alpha_\xi(e^{\pi(X)}) \alpha_\xi(e^{\pi(Y)})} \tag{2.4.6}$$

is analytic on $B_\delta \times B_\delta$. Since LG has a local analytic logarithm and for any $X \in L\mathfrak{g}$, $e^{\pi(X)}$ is a lift of $\pi(\exp_{LG} X)$ by proposition 1.3.2, the local multiplication (2.2.3) is analytic.
(ii) Let ξ as in (i) and $\eta \in \mathcal{H}^{\text{fin}}$. We claim that η is analytic for the canonical action of LG given by $(\gamma, u)\xi = u\xi$. It is sufficient to check this near 1 where we may trivialise $\mathcal{L}G$ using ξ as in §2.2. Then

$$(\gamma, z)\eta = (\gamma, \pi(\gamma)\alpha_\xi(\pi(\gamma))^{-1} z)\eta = \pi(\gamma)\alpha_\xi(\pi(\gamma))^{-1} z\eta \tag{2.4.7}$$

For $\gamma \in LG$ near 1, this is equal to

$$z\alpha_\xi(e^{-\pi(X)}) e^{\pi(X)} \eta \tag{2.4.8}$$

where $X = \log(\gamma)$ and is therefore analytic $\diamond$

PROPOSITION 2.4.3. *Let $(\pi_i, \mathcal{H}_i)$ be positive energy representations of levels ℓ_i. Then,*

(i) *The Lie algebra cocycle corresponding to $\mathcal{L}^i G = \pi_i^* U(\mathcal{H}_i)$ is ℓ_i times the fundamental cocycle*

$$iB(X,Y) = i\int_0^{2\pi} \langle X, \dot{Y}\rangle \frac{d\theta}{2\pi} \tag{2.4.9}$$

(ii) *$\mathcal{L}^1 G$ and $\mathcal{L}^2 G$ are (canonically) isomorphic if, and only if $\ell_1 = \ell_2$.*

PROOF. (i) Fix $\xi \in \mathcal{H}^{\text{fin}}$ and trivialise $\mathcal{L}^i G$ near 1 as in §2.2. We begin by computing the corresponding local adjoint action. By (2.2.3)–(2.2.4)

$$\begin{aligned}(g,z)\star(h,w)\star\mathcal{I}(g,z) &= \left(gh, zw\frac{\alpha_\xi(\pi(g)\pi(h))}{\alpha_\xi(\pi(g))\alpha_\xi(\pi(h))}\right)\star(g^{-1},z^{-1})\\ &= \left(ghg^{-1}, w\frac{\alpha_\xi(\pi(g)\pi(h)\pi(g)^*)}{\alpha_\xi(\pi(h))}\right)\end{aligned} \tag{2.4.10}$$

where the lifts $\pi(g), \pi(h) \in U(\mathcal{H})$ are arbitrary and we have chosen $\pi(g)\pi(h)$ as a lift of $[\pi(gh)]$ and $\pi(g)^*$ as a lift of $[\pi(g)^{-1}]$. Set $h := h_t = \exp_{LG}(tX)$ for some $X \in L\mathfrak{g}$ and choose $e^{t\pi(X)}$ as a lift of $\pi(h_t)$. We will compute the derivative of the right hand-side of (2.4.10) at $t=0$. Since $\xi \in \mathcal{H}^{\text{fin}} \subset \mathcal{H}^\infty$,

$$\left.\frac{d}{dt}\right|_{t=0} e^{t\pi(X)}\xi = \pi(X)\xi \tag{2.4.11}$$

and therefore, assuming that $\|\xi\| = 1$

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_\xi(e^{t\pi(X)}) = \left.\frac{d}{dt}\right|_{t=0} \frac{(e^{t\pi(X)}\xi,\xi)}{\sqrt{(e^{t\pi(X)}\xi,\xi)(\xi,e^{t\pi(X)}\xi)}} = (\pi(X)\xi,\xi) \tag{2.4.12}$$

since, by skew–adjointness, the function at the denominator is even. Similarly, $\pi(g)^*$ leaves $\mathcal{H}^\infty$ invariant by proposition 1.5.3 and therefore

$$\left.\frac{d}{dt}\right|_{t=0} \pi(g)e^{t\pi(X)}\pi(g)^*\xi = \pi(g)\pi(X)\pi(g)^*\xi \tag{2.4.13}$$

whence

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_\xi(\pi(g)e^{t\pi(X)}\pi(g)^*) = (\pi(g)\pi(X)\pi(g)^*\xi,\xi) \tag{2.4.14}$$

Thus

$$\left.\frac{d}{dt}\right|_{t=0} \frac{\alpha_\xi(\pi(g)\pi(h_t)\pi(g)^*)}{\alpha_\xi(\pi(h_t))} = (\pi(g)\pi(X)\pi(g)^*\xi,\xi) - (\pi(X)\xi,\xi) \tag{2.4.15}$$

and the local adjoint action of $\mathcal{L}^i G$ is

$$(g,z)(X,ix)(g,z)^{-1} = (gXg^{-1}, ix + (\pi(g)\pi(X)\pi(g)^*\xi,\xi) - (\pi(X)\xi,\xi)) \tag{2.4.16}$$

where we have identified the Lie algebra of $\mathbb{T}$ with $i\mathbb{R}$. To compute the Lie bracket, set $g := g_s = \exp_{LG}(sY)$, $Y \in L\mathfrak{g}$ and choose the lift $\pi(g_s) = e^{s\pi(Y)}$. Then $\left.\frac{d}{ds}\right|_{s=0} \pi(g_s)^*\xi = -\pi(Y)\xi$ and therefore

$$\left.\frac{d}{ds}\right|_{s=0} (\pi(g_s)\pi(X)\pi(g_s)^*\xi,\xi) = \left.\frac{d}{ds}\right|_{s=0} (\pi(X)\pi(g_s)^*\xi, \pi(g_s)^*\xi) = ([\pi(Y),\pi(X)]\xi,\xi) \tag{2.4.17}$$

and it follows by the commutation relations (1.3.2) that the bracket on the Lie algebra of $\mathcal{L}^i G$ is given by

$$[(X,ix),(Y,iy)] = ([X,Y], i\ell_i B(X,Y) + d\beta_\xi(X,Y)) \tag{2.4.18}$$

where β_ξ is the linear form on $L\mathfrak{g}$ given by $\beta_\xi(Z) = (\pi(Z)\xi,\xi)$ and $d\beta_\xi(X,Y) = \beta_\xi([X,Y])$. Notice that β_ξ is continuous since $\|\pi(Z)\xi\| \leq C|Z|_{\frac{1}{2}}\|\xi\|_1$. Thus, the cocycle classifying the Lie algebra of $\mathcal{L}^i G$ as a central extension of $L\mathfrak{g}$ is $i\ell_i B + d\beta_\xi$ which is cohomologous to $i\ell_i B$.
(ii) follows from (i) and the classification of central extensions of LG [**PS**, 4.4.1 (ii)]. The isomorphism $\mathcal{L}^1 G \cong \mathcal{L}^2 G$ is unique since any two differ by an element of $\text{Hom}(LG,\mathbb{T}) = \{1\}$ [**PS**, 3.4.1] $\diamond$

Remark. Denote by $\widetilde{LG}$ the central extension of LG whose Lie algebra cocycle is $iB(\cdot,\cdot)$. By [**PS**, 4.4.6], $\widetilde{LG}$ is the universal central extension of LG and it is easy to deduce from (i) of proposition 2.4.3 that $\mathcal{L}G \cong \widetilde{LG}/Z_\ell$ where ℓ is the level of $\mathcal{H}$ and $Z_\ell \subset \mathbb{T} \subset \widetilde{LG}$ is the group of ℓ roots of unity. It follows that any positive energy representation of LG at level ℓ gives rise to a continuous, unitary representation of $\widetilde{LG}$ such that the centre of $\widetilde{LG}$ acts by the character $z \rightarrow z^\ell$. Moreover, as conjectured by Pressley and Segal [**PS**, §9.3], $\mathcal{H}$ possesses by theorem 2.4.2 a dense subspace of analytic, and *a fortiori* smooth vector for the action of $\widetilde{LG}$,

CHAPTER III

Fermionic construction of level 1 representations

This chapter is devoted to the well–known *quark model* or Fermionic construction of the level 1 positive energy representations of $L\,\mathrm{Spin}_{2n}$. This parallels the Clifford algebra construction of the spin modules of Spin_{2n} and realises the level 1 representations as summands of two distinct Fermionic Fock spaces, the Ramond and Neveu–Schwarz sectors $\mathcal{F}_R, \mathcal{F}_{NS}$.

Our main interest in this construction comes from Algebraic Quantum Field Theory. We shall prove in chapter IV that the von Neumann algebras generated by the groups $L_I\,\mathrm{Spin}_{2n}$ of loops supported in a proper interval $I \subset S^1$ in positive energy representations are type III_1 factors and satisfy Haag duality in the vacuum sector. These results are necessary for the very definition of fusion and will be derived from their well–known analogues for the free Fermi field. Another feature of the quark model is that the vector primary field is realised as a Fermionic field and therefore obeys L^2 bounds. The continuity properties of other primary fields, most notably the spin ones, are not as easily derived in the fermionic picture and will be proved in chapter VI using the equivalent bosonic construction of level 1 representations of $L\,\mathrm{Spin}_{2n}$.

In section 1, we review the Clifford algebra construction of the SO_{2n} spin modules. We do this in some detail as an explicit description of these modules will be needed elsewhere. In section 2, we construct the level 1 representations of $L\,\mathrm{Spin}_{2n}$ inside the Ramond and Neveu–Schwarz Fock spaces $\mathcal{F}_R, \mathcal{F}_{NS}$ by using a global version of the quark model. In section 3, we identify the abstract action of $L^{\mathrm{pol}}\mathfrak{so}_{2n}$ on $\mathcal{F}_R, \mathcal{F}_{NS}$ corresponding to that of $L\,\mathrm{Spin}_{2n}$ with well–known bilinear expressions in the Fermi field and use this to prove the finite reducibility of $\mathcal{F}_R$ and $\mathcal{F}_{NS}$ under $L\,\mathrm{Spin}_{2n}$. Finally, in section 4, we prove that the vector primary field defines a bounded operator–valued distribution by identifying it with a Fermi field.

1. The finite–dimensional spin representations

We describe below the construction of the SO_{2n} spin modules following Brauer and Weyl's original lines [**BrWe**]. This stems from the simple observation that the Clifford algebra $C(V)$ of $V = \mathbb{R}^{2n}$ is a matrix algebra and therefore possesses a unique irreducible representation $\mathcal{F}$. Conjugating it by the automorphic action of SO_{2n} on $C(V)$ thus leads to unitarily equivalent ones and therefore to a projective representation of SO_{2n} on $\mathcal{F}$. The latter lifts to an ordinary representation of the universal cover Spin_{2n} of SO_{2n} which decomposes as the sum of the two spin modules.

1.1. Clifford algebras and CAR algebras.

Let $V = \mathbb{R}^{2n}$ with inner product $B(\cdot,\cdot)$ and $C(V)$ the corresponding Clifford algebra, that is the complex *–algebra generated by the self–adjoint symbols $\psi(v)$, $v \in V$ satisfying

$$\{\psi(u), \psi(v)\} = \psi(u)\psi(v) + \psi(v)\psi(u) = 2B(u, v) \tag{1.1.1}$$

$C(V)$ is naturally $\mathbb{Z}_2$–graded by decreeing that the $\psi(v)$ are of degree one. We claim that $C(V)$ is isomorphic to the matrix algebra $M_{2^n}(\mathbb{C})$. This may be seen by using the factorisation property $C(V_1 \oplus V_2) = C(V_1)\widehat{\otimes}C(V_2)$, where $\widehat{\otimes}$ is the graded tensor product and the fact that $C(\mathbb{R}^2) = M_2(\mathbb{C})$ but is more conveniently obtained by giving a different presentation of $C(V)$ which naturally suggests a module for it.

Introduce for this purpose an orthogonal complex structure j on V and denote by V_j the corresponding complex vector space with hermitian inner product $(u, v) = B(u, v) + iB(u, jv)$. The *canonical anticommutation relations* or CAR algebra $\mathfrak{A}(V_j)$ is the $\mathbb{Z}_2$–graded, complex $*$–algebra generated by the degree 1, $\mathbb{C}$–linear symbols $c(v)$, $v \in V_j$ subject to the relations

$$\{c(u), c(v)\} = 0 \qquad\qquad \{c(u), c(v)^*\} = (u, v) \tag{1.1.2}$$

These suggest that a natural $\mathfrak{A}(V_j)$–module is the exterior algebra $\Lambda V_j = \bigoplus_{k=0}^n \Lambda^k V_j$ with the $c(v)$ acting as creation operators, namely

$$c(v)\, v_1 \wedge \cdots \wedge v_m = v \wedge v_1 \wedge \cdots \wedge v_m \tag{1.1.3}$$

so that

$$c(v)^*\, v_1 \wedge \cdots \wedge v_m = \sum_{j=1}^m (-1)^{j-1}(v_j, v) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m \tag{1.1.4}$$

Lemma 1.1.1. *The representation of* $\mathfrak{A}(V_j)$ *defined by* (1.1.3)–(1.1.4) *is irreducible and yields an isomorphism* $\mathfrak{A}(V_j) \cong \mathrm{End}(\Lambda V_j)$.

Proof. Let π be the above representation. Notice that the number operator N acting as multiplication by k on $\Lambda^k V_j$ may be written as

$$N = \sum_k c(v_k) c(v_k)^* \tag{1.1.5}$$

where v_k is any complex basis of V_j. Thus, if $T \in \mathrm{End}(\Lambda V_j)$ commutes with $\mathfrak{A}(V_j)$, then $[T, N] = 0$ and T leaves $\Lambda^0 V_j$ invariant. Thus, $T\Omega = \lambda\Omega$ where $\Omega \in \Lambda^0 V_j$ is a generator and therefore $T \equiv \lambda$ by cyclicity of Ω. Thus, π is irreducible and, by the double commutant theorem, $\pi(\mathfrak{A}(V_j)) = \mathrm{End}(\Lambda V_j)$ is of dimension 2^{2n}. Since $\mathfrak{A}(V_j)$ is spanned by the monomials

$$c(v_{i_1}) \cdots c(v_{i_k}) c(v_{\ell_1})^* \cdots c(v_{\ell_m})^* \tag{1.1.6}$$

where $v_1 \dots v_n$ is a complex basis of $V_{\mathbb{C}}$ and $1 \le i_1 < \cdots < i_k \le n$, $1 \le \ell_1 < \cdots < \ell_m \le n$, its dimension is bounded above by 2^{2n} and π is an isomorphism $\diamond$

Returning to the Clifford algebra $C(V)$, we notice that it is canonically isomorphic to $\mathfrak{A}(V_j)$ by

$$\psi(v) \to c(v) + c(v)^* \qquad\qquad c(v) \to \frac{1}{2}\Big(\psi(v) - i\psi(jv)\Big) \tag{1.1.7}$$

and is therefore a matrix algebra whose unique irreducible module may be identified with ΛV_j.

Proposition 1.1.2. *There exists a projective unitary representation* Γ *of* SO_{2n} *on* ΛV_j *satisfying, and uniquely determined by*

$$\Gamma(R)\psi(v)\Gamma(R)^* = \psi(Rv) \tag{1.1.8}$$

and extending the canonical action of the unitary group $U(V_j) \subset \mathrm{SO}_{2n}$ *given by*

$$\Gamma(U)\, v_1 \wedge \cdots \wedge v_k = Uv_1 \wedge \cdots \wedge Uv_k \tag{1.1.9}$$

Moreover, Γ *leaves the even and odd subspaces of* ΛV_j*, namely* $\Lambda_0 V_j = \bigoplus_k \Lambda^{2k} V_j$ *and* $\Lambda_1 V_j = \bigoplus_k \Lambda^{2k+1} V_j$ *invariant.*

Proof. The natural action of SO_{2n} on V induces an automorphic one on $C(V)$ given by $\psi(v) \to \psi(Rv)$, $R \in \mathrm{SO}_{2n}$. Conjugating the representation of $C(V)$ on ΛV_j by $R \in \mathrm{SO}_{2n}$ gives of necessity a unitary equivalent one. It follows that there exists a unitary $\Gamma(R)$ on ΛV_j satisfying (1.1.8). By irreducibility, $\Gamma(R)$ is unique up to a phase so that $\Gamma(R_1)\Gamma(R_2) = \Gamma(R_1 R_2)$ and we get a projective unitary representation of SO_{2n} on ΛV_j. By uniqueness, Γ extends the action of $U(V_j)$. Let $P \in U(V_j)$ be multiplication by -1 on V_j so that $\Lambda_0 V_j, \Lambda_1 V_j$ are the ± 1 eigenspaces of $\Gamma(P)$. Since P commutes with SO_{2n} on V_j, we get $\Gamma(R)\Gamma(P)\Gamma(R)^* = \epsilon(R)\Gamma(P)$ where $\epsilon(R) \in \mathbb{T}$ has square 1. Since SO_{2n} is connected, $\epsilon(R) \equiv \epsilon(1)$ and therefore $\Lambda_0 V_j$ and $\Lambda_1 V_j$ are invariant under SO_{2n} $\diamond$

The projective representation of SO_{2n} on ΛV_j lifts to an ordinary, unitary representation of the universal covering group Spin_{2n} of SO_{2n} which we denote by the same symbol Γ. The lift is unique since any two differ by a character and Spin_{2n} is simple. We will show in §1.3 that $\Lambda_0 V_j$ and $\Lambda_1 V_j$ are irreducible under Spin_{2n} by using the infinitesimal action of $\mathfrak{so}_{2n}$. For this purpose, we give below a more convenient description of ΛV_j borrowed from [**PS**, §12.1].

1.2. Holomorphic spinors.

The complex structure j on V induces a splitting $V_{\mathbb{C}} = V_{\mathbb{C}}^{1,0} \oplus V_{\mathbb{C}}^{0,1}$ of the complexification $V_{\mathbb{C}}$ where the summands are the $\pm i$ eigenspaces of $j \otimes 1$. Both are isotropic for the $\mathbb{C}$–bilinear extension of the inner product B on V, which we denote by the same symbol, since $B(u,v) = B(ju, jv) = -B(u,v)$ if $v \in V_{\mathbb{C}}^{1,0}$ or $V_{\mathbb{C}}^{0,1}$. They are unitarily and anti–unitarily isomorphic to V_j by the maps

$$U_{\pm} : v \to \frac{1}{\sqrt{2}}(v \otimes 1 \mp jv \otimes i) \tag{1.2.1}$$

and we shall identify V_j with $V_{\mathbb{C}}^{1,0}$ via U_+. The latter induces an isomorphism of CAR algebras $\mathfrak{A}(V_j) \cong \mathfrak{A}(V_{\mathbb{C}}^{1,0})$ by $c(f) \to c(U_+ f)$ and a unitary $\Lambda U_+ : \Lambda V_j \to \Lambda V_{\mathbb{C}}^{1,0}$ which is equivariant for the corresponding canonical actions of $\mathfrak{A}(V_j)$ and $\mathfrak{A}(V_{\mathbb{C}}^{1,0})$. Transporting the action of $C(V)$ via ΛU_+, we see by (1.1.7) that $\psi(v)$ acts as

$$c(U_+ v) + c(U_+ v)^* = c(U_+ v) + c(\overline{U_- v})^* \tag{1.2.2}$$

where $\overline{u}$ is the canonical conjugation on $V_{\mathbb{C}}$. We may extend by linearity the definition of the symbols $\psi(v)$, $v \in V$ to $\mathbb{C}$–linear symbols $\psi(v)$, $v \in V_{\mathbb{C}}$ satisfying

$$\{\psi(u), \psi(v)\} = 2B(u,v) \qquad \text{and} \qquad \psi(v)^* = \psi(\overline{v}) \tag{1.2.3}$$

Similarly, we extend the maps $U_{\pm}$ to $\mathbb{C}$–linear maps $V_{\mathbb{C}} \to V_{\mathbb{C}}^{1,0}, V_{\mathbb{C}}^{0,1}$. As is readily verified, $U_{\pm} = \sqrt{2} P_{\pm}$ where the $P_{\pm}$ are the orthogonal projections onto $V_{\mathbb{C}}^{1,0}$ and $V_{\mathbb{C}}^{0,1}$ respectively. It follows from (1.2.2) that the $\psi(v)$, $v \in V_{\mathbb{C}}$ act on $\Lambda V_{\mathbb{C}}^{1,0}$ by $\sqrt{2} c(P_+ v) + \sqrt{2} c(\overline{P_- v})^*$. In particular, since the inner product $(\cdot,\cdot)$ on $V_{\mathbb{C}}$ is given by $(u,v) = B(u, \overline{v})$, we get

$$\psi(v)\, v_1 \wedge \cdots \wedge v_m = \sqrt{2}\, v \wedge v_1 \wedge \cdots \wedge v_m \qquad \text{if } v \in V_{\mathbb{C}}^{1,0} \tag{1.2.4}$$

and

$$\psi(v)\, v_1 \wedge \cdots \wedge v_m = \sum_{j=1}^{m} (-1)^{j-1} \sqrt{2}\, B(v, v_j) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m \qquad \text{if } v \in V_{\mathbb{C}}^{0,1} \tag{1.2.5}$$

Let now $R \in \mathrm{SO}_{2n}$. We give below an explicit formula for the action of $\Gamma(R)$ on $\Lambda V_{\mathbb{C}}^{1,0}$. By the cyclicity of $\Omega \in \Lambda^0 V_{\mathbb{C}}^{1,0}$ under $C(V)$ and (1.1.8), it is sufficient to give a formula for $\Gamma(R)\Omega$. Notice that the action of $R = R \otimes 1$ on $V_{\mathbb{C}}$ leaves B invariant and therefore $RV_{\mathbb{C}}^{0,1}$ is an isotropic subspace for B. Thus, if $U \subseteq RV_{\mathbb{C}}^{0,1}$ is a subspace, the expression

$$\mathrm{Det}(U) = \psi(u_1) \cdots \psi(u_k) \tag{1.2.6}$$

where $u_1 \cdots u_k$ is an orthonormal basis of U is, up to multiplication by a complex number of modulus one, a well–defined element of $C(V)$ by (1.2.3). Let now U be a complementary subspace of $RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1}$ in $RV_{\mathbb{C}}^{0,1}$ and set

$$\mathrm{Det}(RV_{\mathbb{C}}^{0,1} / RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1}) = \mathrm{Det}(U) \tag{1.2.7}$$

This is well–defined only up to elements lying in the two–sided ideal generated by the exterior algebra of $RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1}$. Since the latter annihilates Ω, $\mathrm{Det}(RV_{\mathbb{C}}^{0,1} / RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1})\Omega$ is a well–defined ray in $\Lambda V_{\mathbb{C}}^{1,0}$. We now have

Proposition 1.2.1. *Let $R \in \mathrm{SO}_{2n}$, then*

$$\mathbb{C} \cdot \Gamma(R)\Omega = \mathbb{C} \cdot \mathrm{Det}(RV_{\mathbb{C}}^{0,1} / RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1})\Omega \tag{1.2.8}$$

Proof. The vector $\xi = \Gamma(R)\Omega$ differs from zero and, by (1.1.8) is annihilated by $\psi(RV_{\mathbb{C}}^{0,1})$. Conversely, we claim that any η satisfying these two requirements is a multiple of ξ. Indeed, assuming $\|\xi\| = \|\eta\|$, it is easy to see that the map $x\xi \to x\eta$, $x \in C(V)$ is norm–preserving and therefore well–defined. Since it commutes with $C(V)$, it is a scalar and therefore $\eta = \alpha\xi$, $\alpha \in \mathbb{C}$. We claim now that the right–hand side of (1.2.8) is non–zero. Let $v_1 \dots v_k$ be a basis of a complementary subspace of $RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1}$ in $RV_{\mathbb{C}}^{0,1}$ so that the projections v_i^+ of the v_i on $V_{\mathbb{C}}^{1,0}$ are linearly independent. Then, by (1.2.4)–(1.2.5), the leading term of $\mathrm{Det}(RV_{\mathbb{C}}^{0,1}/RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1})\Omega$ is

$$v_1^+ \wedge \cdots \wedge v_k^+ \Omega \tag{1.2.9}$$

and therefore does not vanish. We check now that $\mathrm{Det}(RV_{\mathbb{C}}^{0,1}/RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1})\Omega$ is annihilated by the elements $\psi(RV_{\mathbb{C}}^{0,1})$. This is clear for $\psi(v)$, $v \in RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1}$ since by isotropy $\psi(v)$ commutes with the Det term and therefore annihilates Ω. If, on the other hand v does not lie in $RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1}$ then we may pick a complementary subspace of $RV_{\mathbb{C}}^{0,1} \cap V_{\mathbb{C}}^{0,1}$ containing v. The Det term may therefore be taken of the form $\psi(v) \wedge \cdots$ and therefore is annihilated by $\psi(v)$. The identity (1.2.8) now follows $\diamond$

1.3. The infinitesimal action of $\mathfrak{so}_{2n}$.

We now characterise Γ infinitesimally. The bilinear form B on $V_{\mathbb{C}}$ yields an isomorphism $\mathfrak{so}_{2n,\mathbb{C}} \cong V_{\mathbb{C}} \wedge V_{\mathbb{C}}$ where the latter acts on $V_{\mathbb{C}}$ by $u \wedge v \;\; w = B(v,w)u - B(u,w)v$ with commutation relations

$$[u_1 \wedge v_1, u_2 \wedge v_2] = (u_1,v_2)v_1 \wedge u_2 + (v_1,u_2)u_1 \wedge v_2 - (u_1,u_2)v_1 \wedge v_2 - (v_1,v_2)u_1 \wedge u_2 \tag{1.3.1}$$

Using this identification, it is easy to see that the map $\rho : \mathfrak{so}_{2n,\mathbb{C}} \to C(V)$

$$u \wedge v \to \frac{1}{4}(\psi(u)\psi(v) - \psi(v)\psi(u)) = \frac{1}{2}(\psi(u)\psi(v) - B(u,v)) \tag{1.3.2}$$

satisfies $[\rho(X),\rho(Y)] = \rho([X,Y])$ for any $X,Y \in \mathfrak{so}_{2n,\mathbb{C}}$ and

$$[\rho(X),\psi(v)] = \psi(Xv) \tag{1.3.3}$$

for any $X \in \mathfrak{so}_{2n,\mathbb{C}}, v \in V_{\mathbb{C}}$. If $X \in \mathfrak{so}_{2n}$, the exponentiation of (1.3.3) in $U(\Lambda V_{\mathbb{C}}^{1,0})$ yields

$$e^{\rho(X)}\psi(v)e^{-\rho(X)} = \psi(e^X v) \tag{1.3.4}$$

and therefore, by irreducibility, $e^{\rho(X)} = \Gamma(e^X)$ in $PU(\Lambda V_{\mathbb{C}}^{1,0})$ so that ρ is the representation of $\mathfrak{so}_{2n}$ corresponding to the projective action of SO_{2n} on $\Lambda V_{\mathbb{C}}^{1,0}$. We now use this to classify $\Lambda_0 V_{\mathbb{C}}^{1,0}$ and $\Lambda_1 V_{\mathbb{C}}^{1,0}$

Proposition 1.3.1. *Let $V_{s_\pm}$ be the irreducible Spin_{2n}–modules with highest weights $s_\pm = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} \pm \theta_n)$. Then,*

$$V_{s_+} \cong \Lambda_\epsilon V_{\mathbb{C}}^{1,0} \qquad \text{and} \qquad V_{s_-} \cong \Lambda_{1-\epsilon} V_{\mathbb{C}}^{1,0} \tag{1.3.5}$$

where $\epsilon = 0$ or 1 according to whether n is even or odd.

Proof. Let $e_1,\dots,e_{2n}$ be an orthonormal basis of V satisfying $je_{2k-1} = e_{2k}$. By (1.2.1), the vectors $f_{\pm k} = \sqrt{2}^{-1}(e_{2k-1} \mp ie_{2k})$, $k = 1 \dots n$ are orthonormal basis of $V_{\mathbb{C}}^{1,0}$ and $V_{\mathbb{C}}^{0,1}$ respectively and $B(f_k,f_l) = \delta_{k+l,0}$. If the maximal torus T of SO_{2n} is chosen as consisting of those elements whose matrices in the basis e_k are of the form

$$\begin{pmatrix} R(t_1) & & 0 \\ & \ddots & \\ 0 & & R(t_n) \end{pmatrix} \qquad \text{with} \qquad R(t_k) = \begin{pmatrix} \cos t_k & -\sin t_k \\ \sin t_k & \cos t_k \end{pmatrix} \tag{1.3.6}$$

the Lie algebra $\mathfrak{t}$ of T has a basis given by

$$\Theta_k = e_{2k} \wedge e_{2k-1} = if_k \wedge f_{-k} \tag{1.3.7}$$

so that Θ_k is a matrix all of whose entries are zero except for the kth diagonal block which is of the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By (1.3.2), Θ_k is represented on $\Lambda V_{\mathbb{C}}^{1,0}$ by $\frac{i}{2}(\psi(f_k)\psi(f_{-k})-1)$ and therefore, by (1.2.4)–(1.2.5),

$$\Theta_k f_{k_1}\wedge\cdots\wedge f_{k_\ell} = \frac{i}{2}\begin{cases} f_{k_1}\wedge\cdots\wedge f_{k_\ell} & \text{if } k\in\{k_1,\ldots,k_\ell\} \\ -f_{k_1}\wedge\cdots\wedge f_{k_\ell} & \text{if } k\notin\{k_1,\ldots,k_\ell\}\end{cases} \tag{1.3.8}$$

so that the vector $f_{k_1}\wedge\cdots\wedge f_{k_\ell}$ corresponds to the weight $\sum_p \theta_{k_p} - \frac{1}{2}\sum_k \theta_k$ where the θ_k are the dual basis elements to the $-i\Theta_k$. It follows that the part of $\Lambda V_{\mathbb{C}}^{1,0}$ containing the top exterior power $\Lambda^n V_{\mathbb{C}}^{1,0} = \mathbb{C}f_1\wedge\cdots\wedge f_n$ has highest weight s_+ and the other has highest weight s_- with corresponding weight vector $f_1\wedge\cdots\wedge f_{n-1}$. Thus, $V_{s_+}\subset\Lambda_\epsilon V_{\mathbb{C}}^{1,0}$ and $V_{s_-}\subset\Lambda_{1-\epsilon}V_{\mathbb{C}}^{1,0}$. To conclude, notice that the parity subspaces of $\Lambda V_{\mathbb{C}}^{1,0}$ are irreducible under the even part of the Clifford algebra and therefore $\mathfrak{so}_{2n}$ since the former is generated by the $\psi(u)\psi(v)$ $\diamond$

Remark. By the tensor product rules of proposition I.2.2.2, $\mathrm{Hom}_{\mathrm{Spin}_{2n}}(V_{\mathbb{C}}\otimes V_{s_\pm}, V_{s_\mp})\cong\mathbb{C}$. The corresponding intertwiners may be constructed using the Clifford multiplication map

$$V_{\mathbb{C}}\otimes\Lambda V_{\mathbb{C}}^{1,0}\rightarrow\Lambda V_{\mathbb{C}}^{1,0} \qquad\qquad v\otimes w\rightarrow\psi(v)w \tag{1.3.9}$$

which, by (1.3.3) commutes with Spin_{2n}. Its restrictions to $V_{\mathbb{C}}\otimes\Lambda_0 V_{\mathbb{C}}^{1,0}$ and $V_{\mathbb{C}}\otimes\Lambda_1 V_{\mathbb{C}}^{1,0}$ are the required intertwiners.

2. The infinite–dimensional spin representations

In this section, we construct the level 1 representations of $L\,\mathrm{Spin}_{2n}$ by using an analogue of the Clifford algebra construction of section 1 and realise them, grouped in pairs as summands of two distinct exterior algebras or Fermionic Fock spaces, the Neveu–Schwarz and Ramond sectors $\mathcal{F}_R, \mathcal{F}_{NS}$.

We begin by discussing the representations of the Clifford algebra $C(\mathcal{H})$ of a real, infinite–dimensional Hilbert space. As in section 1, these may be obtained by introducing an orthogonal complex structure J on $\mathcal{H}$ and regarding the latter as a complex Hilbert space $\mathcal{H}_J$. An irreducible representation of $C(\mathcal{H})$ is then got via the canonical action of the isomorphic CAR algebra $\mathfrak{A}(\mathcal{H}_J)$ on the Fock space $\Lambda\mathcal{H}_J$. Unlike the finite–dimensional case however, different complex structures J_1, J_2 lead in general to inequivalent representations and we give below a necessary and sufficient criterion due to I. Segal [**BSZ**] for that to be the case. The discussion is technically simpler when J_1 and J_2 commute with a given complex structure i on $\mathcal{H}$ and we first formulate the criterion as the equivalence of representations of the reference CAR algebra $\mathfrak{A}(\mathcal{H}_i)$. This is the context of *complex fermions* and is treated in §2.1. The more general case of *real fermions* is studied in §2.3.

The complex Fermionic criterion leads at once to a projective representation of a distinguished subgroup of the unitary group of $\mathcal{H}_i$ on the Fock space $\Lambda\mathcal{H}_J$. In §2.2, we derive from it the *basic representation* of the loop group of U_n starting from its unitary action on $L^2(S^1,\mathbb{C}^n)$, as done in [**Wa3**]. Similarly, the real Fermionic criterion yields a projective representation of a subgroup of the orthogonal group of $\mathcal{H}$ extending the previous one. When applied to the orthogonal action of $L\,\mathrm{SO}_{2n}$ on the Hilbert spaces of periodic and anti–periodic functions on S^1 with values in $\mathbb{R}^{2n}$, this yields positive energy representations of $L\,\mathrm{Spin}_{2n}$ and is carried out in §2.4. These are classified in §2.3 and shown to contain all level 1 representations.

2.1. Complex fermions and CAR algebras.

Consider a *complex*, infinite–dimensional Hilbert space $\mathcal{H}$ with hermitian form $(\cdot,\cdot)$ and the corresponding CAR algebra $\mathfrak{A}(\mathcal{H})$ defined as in section 1 by the $\mathbb{C}$–linear symbols $c(f)$, $f\in\mathcal{H}$ subject to the anticomutation relations

$$\{c(f),c(g)\}=0 \qquad\qquad \{c(f),c(g)^*\}=(f,g) \tag{2.1.1}$$

$\mathfrak{A}(\mathcal{H})$ possesses a canonical representation on the exterior algebra or Fermionic Fock space $\Lambda\mathcal{H}$ given by

$$c(f)\, g_1 \wedge \cdots \wedge g_m = f \wedge g_1 \wedge \cdots \wedge g_m \tag{2.1.2}$$

so that

$$c(f)^*\, g_1 \wedge \cdots \wedge g_m = \sum_{j=1}^{m} (-1)^{j-1} (g_j, f) g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge g_m \tag{2.1.3}$$

which is irreducible [**BSZ**, cor. 3.3.1]. We also consider the underlying real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with inner product $B(\cdot,\cdot) = \Re(\cdot,\cdot)$ and the associated Clifford algebra $C(\mathcal{H}_{\mathbb{R}})$ generated by the real–linear, self–adjoint symbols $\psi(f)$, $f \in \mathcal{H}_{\mathbb{R}}$ satisfying

$$\{\psi(f), \psi(g)\} = 2B(f,g) \tag{2.1.4}$$

The maps

$$\psi(f) \to c(f) + c(f)^* \qquad\qquad c(f) \to \frac{1}{2}\Big(\psi(f) - i\psi(if)\Big) \tag{2.1.5}$$

extend to an isomorphism $C(\mathcal{H}_{\mathbb{R}}) \cong \mathfrak{A}(\mathcal{H})$ which we shall however use in a somewhat opposite way to that of section 1. Indeed, choosing an orthogonal complex structure J on $\mathcal{H}_{\mathbb{R}}$ differing from the original one and denoting by $\mathcal{H}_J$ the corresponding complex Hilbert space, we get $\mathfrak{A}(\mathcal{H}) \cong C(\mathcal{H}_{\mathbb{R}}) \cong \mathfrak{A}(\mathcal{H}_J)$ and therefore a representation π_J of $\mathfrak{A}(\mathcal{H})$ via the canonical action of $\mathfrak{A}(\mathcal{H}_J)$ on the Fock space $\mathcal{F}_J = \Lambda\mathcal{H}_J$.

The equivalence class of π_J depends upon J in a way explained below. The discussion is simpler if J commutes with the original complex structure on $\mathcal{H}$ so that it is unitary. We may then diagonalise it and write $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where the $\mathcal{H}_\pm$ are the $\pm i$ eigenspaces of J. Then, $\mathcal{H}_J = \mathcal{H}_+ \oplus \overline{\mathcal{H}_-}$ where $\overline{\mathcal{H}_-} \cong \mathcal{H}_-^*$ is $\mathcal{H}_-$ with a reversed complex structure and $\mathcal{F}_J = \Lambda\mathcal{H}_+ \widehat{\otimes} \Lambda\mathcal{H}_-^*$. The corresponding action of $\mathfrak{A}(\mathcal{H})$ on $\mathcal{F}_J$ is explicitly given by

$$\pi_J(c(f)) = c(f_+) + c(f_-)^* \tag{2.1.6}$$

where $f_\pm$ are the projections of f on $\mathcal{H}_\pm$.

In marked contrast to the finite–dimensional case, the automorphic action of the full unitary group of $\mathcal{H}$ is not implemented on $\mathcal{F}_J$ in the sense that, for a given $u \in U(\mathcal{H})$ there does not exist in general a unitary operator $\Gamma(u)$ on $\mathcal{F}_J$ such that $\Gamma(u)\pi_J(c(f))\Gamma(u)^* = \pi_J(c(uf))$. Notice though that the operator $\Gamma(u)$, when it exists is unique up to a phase since π_J is irreducible. The existence of $\Gamma(u)$ is in fact easily seen to be equivalent to the unitary equivalence of π_J and $\pi_{\widetilde{J}}$ where $\widetilde{J} = uJu^*$. The equivalence criterion given below determines a projective representation of a distinguished subgroup of $U(\mathcal{H})$ which we presently define. Consider the *restricted unitary group* of $\mathcal{H}$, relative to the unitary complex structure J given by

$$U_{\text{res}}(\mathcal{H}, J) = \{u \in U(\mathcal{H}) |\; \|[u,J]\|_{\text{HS}} < \infty\} \tag{2.1.7}$$

where $\|\cdot\|_{\text{HS}}$ is the Hilbert–Schmidt norm. We topologise $U_{\text{res}}(\mathcal{H}, J)$ by endowing it with the strong operator topology combined with the metric topology determined by $d(u,v) = \|[u-v, J]\|_{\text{HS}}$. It contains the intersection of the unitary groups of $\mathcal{H}$ and $\mathcal{H}_J$. The automorphic action of the latter is implemented on $\mathcal{F}_J$ by

$$\Gamma(u) g_1 \wedge \cdots \wedge g_m = ug_1 \wedge \cdots \wedge ug_m \tag{2.1.8}$$

we refer to this as the *canonical quantisation* of $U(\mathcal{H}) \cap U(\mathcal{H}_J)$. The following result is due to I. Segal [**BSZ**]. A short proof as well as the details of (iii), may be found in [**Wa3**].

THEOREM 2.1.1. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathfrak{A}(\mathcal{H})$ the corresponding CAR algebra. Then*

(i) *Any unitary complex structure J on $\mathcal{H}$ determines by (2.1.6) an irreducible representation π_J of $\mathfrak{A}(\mathcal{H})$ on $\mathcal{F}_J = \Lambda\mathcal{H}_+ \widehat{\otimes} \Lambda\overline{\mathcal{H}}_-$ where the $\mathcal{H}_\pm$ are the $\pm i$ eigenspace of J.*

(ii) *The representations π_{J_1} and π_{J_2} are unitarily equivalent if, and only if $J_1 - J_2$ is a Hilbert–Schmidt operator.*

(iii) *For any unitary complex structure J, there exists a strongly continuous, projective unitary representation Γ of the restricted unitary group* (2.1.7) *on $\mathcal{F}_J$ satisfying and uniquely determined by*

$$\Gamma(u)\pi_J(c(f))\Gamma(u)^* = \pi_J(c(uf)) \tag{2.1.9}$$

2.2. The basic representation of LU_n.

Let $LU_n = C^\infty(S^1, U_n)$ and consider the action of $LU_n \rtimes \mathrm{Rot}\, S^1$ on the Hilbert space $\mathcal{H} = L^2(S^1, \mathbb{C}^n)$ where LU_n acts by multiplication and $\mathrm{Rot}\, S^1$ by $R_\theta f = f_\theta$ where $f_\theta(\phi) = f(\phi - \theta)$. We denote the standard complex structure on $\mathcal{H}$ by i. Thus, $R_\theta = e^{i\theta d}$ where the self–adjoint generator of rotations $d = i\frac{d}{d\theta}$ is non–negative on the subspace $\mathcal{H}_+$ spanned by the functions $e^{im\theta} \otimes v$, $v \in \mathbb{C}^n$, $m \leq 0$ and negative on $\mathcal{H}_- = \mathcal{H}_+^\perp$. Let J be the complex structure acting as $\pm i$ on $\mathcal{H}_\pm$. Up to a finite–dimensional ambiguity on the space of constant functions $\mathbb{C}^n \subset \mathcal{H}$, J is the unique complex structure on $\mathcal{H}$ such that R_θ is a unitary action on $\mathcal{H}_J$ decomposing into a sum of non–negative characters. In other words, J commutes with R_θ and the corresponding self–adjoint generator on $\mathcal{H}_J$, namely $J\frac{d}{d\theta}$ is non–negative.

Consider the representation π_J of the CAR algebra $\mathfrak{A}(\mathcal{H})$ on $\mathcal{F} = \Lambda\mathcal{H}_J = \Lambda\mathcal{H}_+ \widehat{\otimes} \Lambda\overline{\mathcal{H}_-}$ given as in (2.1.6). R_θ commutes with J and is therefore canonically quantised on $\mathcal{F}$, call the corresponding action $U_\theta = \Gamma(R_\theta)$. Since R_θ has non–negative spectrum and finite–dimensional eigenspaces on $\mathcal{H}_J$, U_θ is a positive energy representation of $\mathrm{Rot}\, S^1$. Let $\gamma \in LU_n$ with Fourier expansion $\sum_{m\in\mathbb{Z}} \widehat{\gamma}(m) e^{im\theta}$. It is readily verified, using the basis $e^{im\theta} \otimes v$ of $\mathcal{H}$ that

$$\|[\gamma, J]\|_{\mathrm{HS}}^2 = 4\sum_m |m| \|\widehat{\gamma}(m)\|^2 \tag{2.2.1}$$

where $\|\widehat{\gamma}(m)\|$ is the Hilbert–Schmidt norm on $\mathrm{End}(\mathbb{C}^n)$. On the other hand, the operator norm of γ is equal to the supremum norm $\|\gamma\|_\infty$ and by theorem 2.1.1 we get a continuous projective unitary action of LU_n of positive energy on $\mathcal{F}$ called the *basic representation* of LU_n.

2.3. Real fermions and Clifford algebras.

Let now $\mathcal{H}$ be a *real* Hilbert space with inner product $B(\cdot, \cdot)$ and Clifford algebra $C(\mathcal{H})$. Every orthogonal complex structure J on $\mathcal{H}$ gives rise to a representation π_J of $C(\mathcal{H})$ on the exterior algebra $\Lambda\mathcal{H}_J$ via the isomorphism $C(\mathcal{H}) \cong \mathfrak{A}(\mathcal{H}_J)$. Explicitly,

$$\pi_J(\psi(f)) = c(f) + c(f)^* \tag{2.3.1}$$

where $c(f)$ and $c(f)^*$ act as in (2.1.2)–(2.1.3). We give below a criterion for the representations of $C(\mathcal{H})$ determined by two orthogonal complex structures J_1, J_2 to be unitary equivalent. Its proof proceeds by noting that, up to the replacement of $\mathcal{H}$ by $\mathcal{H} \oplus \mathcal{H} = \mathcal{H} \otimes \mathbb{C}$ and J_k by $J_k \otimes 1$, one may always assume that J_1, J_2 commute with a reference complex structure on $\mathcal{H}$. The criterion is then obtained from that of theorem 2.1.1 in view of the following simple observation

Lemma 2.3.1. *Let $\mathcal{H}$ be a real Hilbert space with Clifford algebra $C(\mathcal{H})$ and J_1, J_2 two orthogonal complex structures. Then, the representations of $C(\mathcal{H})$ on $\Lambda\mathcal{H}_{J_1}$, $\Lambda\mathcal{H}_{J_2}$ are unitarily equivalent if, and only if those of $C(\mathcal{H} \oplus \mathcal{H})$ on $\Lambda(\mathcal{H}_{J_1} \oplus \mathcal{H}_{J_1})$, $\Lambda(\mathcal{H}_{J_2} \oplus \mathcal{H}_{J_2})$ are.*

Proof. We proceed as in [**SS**, lemma 1]. Let $V : \Lambda\mathcal{H}_{J_1} \to \Lambda\mathcal{H}_{J_2}$ be a unitary $C(\mathcal{H})$–intertwiner. Then,

$$V \otimes V : \Lambda(\mathcal{H}_{J_1} \oplus \mathcal{H}_{J_1}) \cong \Lambda\mathcal{H}_{J_1} \widehat{\otimes} \Lambda\mathcal{H}_{J_1} \longrightarrow \Lambda\mathcal{H}_{J_2} \widehat{\otimes} \Lambda\mathcal{H}_{J_2} \cong \Lambda(\mathcal{H}_{J_2} \oplus \mathcal{H}_{J_2}) \tag{2.3.2}$$

intertwines the action of $C(\mathcal{H} \oplus \mathcal{H}) = C(\mathcal{H}) \widehat{\otimes} C(\mathcal{H})$. Conversely, let $U : \Lambda\mathcal{H}_{J_1} \widehat{\otimes} \Lambda\mathcal{H}_{J_1} \to \Lambda\mathcal{H}_{J_2} \widehat{\otimes} \Lambda\mathcal{H}_{J_2}$ be a $C(\mathcal{H}) \widehat{\otimes} C(\mathcal{H})$–intertwiner and $\Omega_1 \otimes \Omega_1 \in \Lambda\mathcal{H}_{J_1} \widehat{\otimes} \Lambda\mathcal{H}_{J_1}$ be the vacuum vector. We claim that $U(\Omega_1 \otimes \Omega_1) = \xi \otimes \eta$ for some $\xi, \eta \in \Lambda\mathcal{H}_{J_2}$ of norm one. Assuming this for the moment, one sees that the map $V : \Lambda\mathcal{H}_{J_1} \to \Lambda\mathcal{H}_{J_2}$ given by $\pi_{J_1}(x)\Omega_1 \to \pi_{J_2}(x)\xi$ is well–defined and norm preserving since, for any $x \in C(\mathcal{H})$

$$\|\pi_{J_2}(x)\xi\| = \|\pi_{J_2}(x) \otimes 1(\xi \otimes \eta)\| = \|U(\pi_{J_1}(x) \otimes 1)(\Omega_1 \otimes \Omega_1)\| = \|\pi_{J_1}(x)\Omega_1\| \tag{2.3.3}$$

and therefore yields a unitary map intertwining $C(\mathcal{H})$. Returning to our claim, let $U(\Omega_1 \otimes \Omega_1) = \sum_k \xi_k \otimes \eta_k$ where $(\eta_k, \eta_{k'}) = \delta_{k,k'}$. Then,

$$\begin{aligned}(\pi_{J_1}(x)\Omega_1, \Omega_1) &= (\pi_{J_1}(x) \otimes 1(\Omega_1 \otimes \Omega_1), \Omega_1 \otimes \Omega_1) \\ &= (U\pi_{J_1}(x) \otimes 1U^*U(\Omega_1 \otimes \Omega_1), U(\Omega_1 \otimes \Omega_1)) \\ &= \sum_k (\pi_{J_2}(x)\xi_k, \xi_k)\end{aligned} \tag{2.3.4}$$

However, π_{J_1} is the GNS representation of $C(\mathcal{H}) \cong \mathfrak{A}(\mathcal{H}_{J_1})$ for the vector state $\phi(x) = (\pi_{J_1}(x)\Omega_1, \Omega_1)$. By theorem 2.1.1, π_{J_1} is irreducible so that ϕ is pure. It follows that all non–zero states $(\pi_{J_2}(x)\xi_k, \xi_k)$ must be equal. By irreducibility of π_{J_2}, this is the case if, and only if all ξ_k are proportional and therefore $U\Omega_1 \otimes \Omega_1 = \xi \otimes \eta$ as claimed $\diamond$

It follows almost at once that π_{J_1} is unitarily equivalent to π_{J_2} if, and only if $J_1 - J_2$ is a (real) Hilbert–Schmidt operator on $\mathcal{H}$. As in theorem 2.1.1, this criterion leads to a projective representation of a distinguished subgroup of the orthogoanl group $O(\mathcal{H})$. More precisely, consider the *restricted orthogonal group* of $\mathcal{H}$ relative to J defined by

$$O_{\text{res}}(\mathcal{H}, J) = \{R \in O(\mathcal{H}) |\, \|[R, J]\|_{\text{HS}} < \infty\} \tag{2.3.5}$$

where the Hilbert–Schmidt norm refers to $[R, J]$ as a real operator. We topologise $O_{\text{res}}(\mathcal{H}, J)$ by the strong operator topology combined with the metric $d(R, S) = \|[R-S, J]\|_{HS}$. This gives, for any other reference complex structure i on $\mathcal{H}$ commuting with J a continuous inclusion $U_{\text{res}}(\mathcal{H}_i, J) \subset O_{\text{res}}(\mathcal{H}, J)$. The following theorem is due to Shale and Stinespring [**BSZ, SS**]

THEOREM 2.3.2. *Let $\mathcal{H}$ be a real Hilbert space with Clifford algebra $C(\mathcal{H})$. Then*

(i) *Any orthogonal complex structure J on $\mathcal{H}$ determines an irreducible representation π_J of $C(\mathcal{H})$ on $\mathcal{F}_J = \Lambda\mathcal{H}_J$ given by*

$$\pi_J(\psi(f)) = c(f) + c(f)^* \tag{2.3.6}$$

(ii) *The representations π_{J_1} and π_{J_2} are unitarily equivalent if, and only if $J_1 - J_2$ is a Hilbert–Schmidt operator.*

(iii) *For any unitary J, there exists a strongly continuous, projective unitary representation Γ of the restricted orthogonal group* (2.3.5) *on $\mathcal{F}_J$ satisfying and uniquely determined by*

$$\Gamma(R)\pi_J(\psi(f))\Gamma(R)^* = \pi_J(\psi(Rf)) \tag{2.3.7}$$

(iv) *If J commutes with a reference complex structure i on $\mathcal{H}$, Γ extends the representation of $U_{\text{res}}(\mathcal{H}_i, J) \subset O_{\text{res}}(\mathcal{H}, J)$ given by theorem 2.1.1.*

PROOF. (i) follows at once from the isomorphism $C(\mathcal{H}) \cong \mathfrak{A}(\mathcal{H}_J)$ given by (2.1.5) and (i) of theorem 2.1.1.
(ii) By lemma 2.3.1, $\Lambda\mathcal{H}_{J_1}$ and $\Lambda\mathcal{H}_{J_2}$ are unitarily equivalent as $C(\mathcal{H})$–modules iff $\Lambda(\mathcal{H}_{J_k} \oplus \mathcal{H}_{J_k})$, $k = 1, 2$ are unitarily equivalent as $C(\mathcal{H} \oplus \mathcal{H})$–modules. On the other hand, $\mathcal{H} \oplus \mathcal{H} = \mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}$ possesses a natural complex structure commuting with $J_k \otimes 1$ and since $C(\mathcal{H} \oplus \mathcal{H}) \cong \mathfrak{A}(\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C})$, theorem 2.1.1 implies that the original representations are equivalent if, and only $J_1 \otimes 1 - J_2 \otimes 1$ is a complex Hilbert–Schmidt operator on $\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}$ and therefore if, and only if $J_1 - J_2$ is a real Hilbert–Schmidt operator on $\mathcal{H}$.
(iii) Let $R \in O(\mathcal{H})$ and regard it as a unitary operator $\mathcal{H}_J \to \mathcal{H}_{\widetilde{J}}$ where $\widetilde{J} = RJR^{-1}$. As such it extends to a unitary $\Lambda R : \Lambda\mathcal{H}_J \to \Lambda\mathcal{H}_{\widetilde{J}}$ satisfying $\Lambda R\pi_J(c(f))\Lambda R^* = \pi_{\widetilde{J}}(c(Rf))$. Notice that $[R, J]$ is Hilbert–Schmidt if, and only if $J - \widetilde{J}$ is. Thus, if $R \in O_{\text{res}}(\mathcal{H}, J)$, there exists by (i) a unitary $V : \Lambda\mathcal{H}_J \to \Lambda\mathcal{H}_{\widetilde{J}}$ intertwining $C(\mathcal{H})$. The operator $\Gamma(R) = V^*RV$ is easily seen to have all the required properties. Moreover, by irreducibility, $\Gamma(R)$ is uniquely determined, up to a phase by (2.3.7) and in particular, $\Gamma(R_1R_2) = \Gamma(R_1)\Gamma(R_2)$. As is readily verified, Γ is such that the following diagram

commutes

$$\begin{tikzcd} O_{\mathrm{res}}(\mathcal{H},J) \arrow[r] \arrow[dr, "\Gamma"'] & U_{\mathrm{res}}(\mathcal{H}\otimes_{\mathbb{R}}\mathbb{C}, J\otimes 1) \arrow[r, "\Gamma_\otimes"] & PU(\Lambda\mathcal{H}_J\widehat{\otimes}\Lambda\mathcal{H}_J) \\ & PU(\Lambda\mathcal{H}_J) \arrow[ur, "\Delta"'] & \end{tikzcd} \tag{2.3.8}$$

where $\Gamma_\otimes$ is the representation of $U_{\mathrm{res}}(\mathcal{H}\otimes_{\mathbb{R}}\mathbb{C}, J\otimes 1)$ given by theorem 2.1.1 and $\Delta(u) = u\otimes u$. It follows that Γ is continuous since Δ is a homeomorphism onto its image.
(iv) is clear in view of the uniqueness of the operators $\Gamma(R)$, (2.1.9) and the fact that the isomorphism $C(\mathcal{H}) \cong \mathfrak{A}(\mathcal{H}_i)$ is equivariant for the automorphic actions of $U(\mathcal{H}_i) \subset O(\mathcal{H})$ $\diamond$

2.4. The Ramond and Neveu–Schwarz Fock spaces.

We construct below positive energy representations of $L\,\mathrm{Spin}_{2n}$ by mapping it to the restricted orthogonal groups of suitable real Hilbert spaces and using theorem 2.3.2. These representations factor through the loop group of SO_{2n} and extend the basic representation of $LU_n \subset L\,\mathrm{SO}_{2n}$ obtained in §2.2.

Consider the Ramond and Neveu–Schwarz Hilbert spaces $\mathcal{H}_{\mathrm{R}}, \mathcal{H}_{\mathrm{NS}}$ of square integrable, $\mathbb{C}^n$–valued periodic and anti–periodic functions on S^1 spanned by the functions $e^{im\theta}\otimes v$ with $m\in\mathbb{Z}$ or $m\in\frac{1}{2}+\mathbb{Z}$ respectively. They are naturally complex Hilbert spaces with standard complex structure i but we shall regard them as real ones. As such, they support orthogonal actions of $L\,\mathrm{SO}_{2n}\rtimes\mathrm{Rot}\,S^1$ where the first factor acts by multiplication on $\mathbb{C}^n\cong\mathbb{R}^{2n}$–valued functions and the second by $R_\theta f = f_\theta$ [1]. Let $\mathcal{H}$ be $\mathcal{H}_{\mathrm{R}}$ or $\mathcal{H}_{\mathrm{NS}}$ and split is as a direct sum $\mathcal{H}_+\oplus\mathcal{H}_-$ where the $\mathcal{H}_\pm$ are the (complex) subspaces spanned by the functions $e^{im\theta}\otimes v$, $m\leq 0$ and $m>0$ respectively with $m\in\mathbb{Z}$ for $\mathcal{H}_{\mathrm{R}}$ and $m\in\frac{1}{2}+\mathbb{Z}$ for $\mathcal{H}_{\mathrm{NS}}$. Let J be the complex structure acting as multiplication by $\pm i$ on $\mathcal{H}_\pm$. If $\mathcal{H}$ is endowed with the complex structure J, the action of $\mathrm{Rot}\,S^1$ is unitary and has non–negative spectrum and finite–dimensional eigenspaces. Its canonical quantisation on $\mathcal{F}=\Lambda\mathcal{H}_J$ is therefore of positive energy. Let $O_{\mathrm{res}}(\mathcal{H},J)$ be defined by (2.3.5), then

LEMMA 2.4.1. *Let $\mathcal{H}=\mathcal{H}_{\mathrm{R}}$ or $\mathcal{H}_{\mathrm{NS}}$ and J the complex structure defined above. Then, $L\,\mathrm{SO}_{2n}\subset O_{\mathrm{res}}(\mathcal{H},J)$ and the inclusion is continuous.*

PROOF. Let $\gamma\in L\,\mathrm{SO}_{2n}$. To compute $\|[\gamma,J]\|_{\mathrm{HS}}$, it is more convenient to complexify $\mathcal{H}$ and consider the splitting

$$\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}}^{1,0}\oplus\mathcal{H}_{\mathbb{C}}^{0,1} \tag{2.4.1}$$

where $\mathcal{H}_{\mathbb{C}}^{1,0}$, $\mathcal{H}_{\mathbb{C}}^{0,1}$ are the $\pm 1\otimes i$ eigenspaces of $J\otimes 1$ on $\mathcal{H}_{\mathbb{C}}$. If $\mathcal{H}=\mathcal{H}_{\mathrm{NS}}$, these are spanned by the functions $v(m)=e^{(1\otimes i)m\theta}v$, $v\in V_{\mathbb{C}}$ and $m\lessgtr 0$ respectively. For $\mathcal{H}=\mathcal{H}_{\mathrm{R}}$, $\mathcal{H}_{\mathbb{C}}^{1,0}$ is spanned by the $v(m)$ with $m<0$ or $m=0$ and $v\in V_{\mathbb{C}}^{1,0}$, the $1\otimes i$ eigenspace of $i\otimes 1$ on $V_{\mathbb{C}}$ while $\mathcal{H}_{\mathbb{C}}^{0,1}$ is spanned by $v(m)$ with $m>0$ or $m=0$ and $v\in V_{\mathbb{C}}^{0,1}$, the $-1\otimes i$ eigenspace of $i\otimes 1$ on $V_{\mathbb{C}}$. Using these basis, a simple computation shows that for $\gamma\in L\,\mathrm{SO}_{2n}$,

$$\|[\gamma\otimes 1, J\otimes 1]\|^2_{\mathrm{HS}(\mathcal{H}_{\mathbb{C}})}\leq 4\sum_{m\in\mathbb{Z}}(|m|+1)\|\widehat{\gamma}(m)\|^2 \tag{2.4.2}$$

where the $\widehat{\gamma}(m)\in\mathrm{End}(V_{\mathbb{C}})$ are the Fourier coefficients of γ and $\|\widehat{\gamma}(m)\|$ is the Hilbert–Schmidt norm on $\mathrm{End}(V_{\mathbb{C}})$. The lemma follows since $\|[\gamma,J]\|_{\mathrm{HS}(\mathcal{H})}=\|[\gamma\otimes 1,J\otimes 1]\|_{\mathrm{HS}(\mathcal{H}_{\mathbb{C}})}$ $\diamond$

By lemma 2.4.1 and theorem 2.3.2, we get a continuous projective representation of $L\,\mathrm{SO}_{2n}$ on the Fock space $\mathcal{F}=\Lambda\mathcal{H}_J$ with $\mathcal{H}=\mathcal{H}_{\mathrm{R}}$ or $\mathcal{H}_{\mathrm{NS}}$ which, as remarked above is of positive energy. Notice that by identifying U_n with the subgroup of SO_{2n} commuting with the complex structure on $\mathbb{R}^{2n}\cong\mathbb{C}^n$, we get a natural inclusion $LU_n\subset L\,\mathrm{SO}_{2n}$. The restriction of $\mathcal{H}_{\mathrm{R}}$ to LU_n is the standard representation of LU_n on $L^2(S^1,\mathbb{C}^n)$ considered in §2.2, and the restriction of $\mathcal{H}_{\mathrm{NS}}$ to LU_n is unitary equivalent to $L^2(S^1,\mathbb{C}^n)$ by the map $f\to e^{-i\frac{\theta}{2}}f$. Since these identifications carry the complex structure J on $\mathcal{H}_{\mathrm{R}}$

[1] The action of $\mathrm{Rot}\,S^1$ on $\mathcal{H}_{\mathrm{NS}}$ is two–valued and is therefore really one of the non–trivial double cover of $\mathrm{Rot}\,S^1$.

or $\mathcal{H}_{\mathrm{NS}}$ considered above to the one on $L^2(S^1,\mathbb{C}^n)$ given in §2.2, the restrictions of $\mathcal{F}_R,\mathcal{F}_{NS}$ to LU_n are equivalent to its basic representation.

Lemma 2.4.2. *Let $\mathcal{H}=\mathcal{H}_{\mathrm{R}}$ or $\mathcal{H}_{\mathrm{NS}}$ and $\mathcal{F}_0=\bigoplus_{k\geq 0}\Lambda^{2k}\mathcal{H}_J$ and $\mathcal{F}_1=\bigoplus_{k\geq 0}\Lambda^{2k+1}\mathcal{H}_J$ the even and odd subspaces of $\mathcal{F}$. Then, each is invariant under $L\,\mathrm{Spin}_{2n}\rtimes\mathrm{Rot}\,S^1$ and therefore defines a positive energy representation of $L\,\mathrm{Spin}_{2n}$.*

Proof. Let P be multiplication by -1 on $\mathcal{H}$. P is canonically quantised on $\mathcal{F}=\Lambda\mathcal{H}_J$ and $\mathcal{F}_0,\mathcal{F}_1$ are the ± 1 eigenspaces of $\Gamma(P)$. The action of $\mathrm{Rot}\,S^1$ on $\mathcal{H}$ commutes with P and J. It is canonically quantised on $\mathcal{F}$ and commutes with $\Gamma(P)$ and therefore leaves each $\mathcal{F}_k$ invariant. $L\,\mathrm{Spin}_{2n}$ commutes with P on $\mathcal{H}$ hence

$$\Gamma(\gamma)\Gamma(P)\Gamma(\gamma)^*=\epsilon(\gamma)\Gamma(P) \tag{2.4.3}$$

for some $\epsilon(\gamma)\in\mathbb{T}$ depending continuously on γ. Since $L\,\mathrm{Spin}_{2n}$ is connected and $\epsilon(\gamma)^2=1$, we have $\epsilon(\gamma)=\epsilon(1)=1$ so that the $\mathcal{F}_k$ are invariant under $L\,\mathrm{Spin}_{2n}$. We now distinguish two cases. If $\mathcal{H}=\mathcal{H}_{\mathrm{R}}$, the action of $\mathrm{Rot}\,S^1$ on $\mathcal{H}$ is a genuine one and we therefore get a positive energy representation of $L\,\mathrm{Spin}_{2n}\rtimes\mathrm{Rot}\,S^1$ on $\mathcal{F}_0,\mathcal{F}_1$. On the other hand, the action of $\mathrm{Rot}\,S^1$ on $\mathcal{H}=\mathcal{H}_{\mathrm{NS}}$ is really one of its non–trivial double cover $\widetilde{\mathrm{Rot}\,S^1}$. This leads to a positive energy representation of $\widetilde{\mathrm{Rot}\,S^1}$ on $\mathcal{F}$ with characters $z\rightarrow z^n$, $n\in\mathbb{N}$ on $\mathcal{F}^0$ and z^n, $n\in\frac{1}{2}+\mathbb{N}$ on $\mathcal{F}^1$. Thus, multiplying the action on $\mathcal{F}^1$ by the character $z^{-\frac{1}{2}}$ we get a positive energy representation of $\mathrm{Rot}\,S^1$ on $\mathcal{F}_0,\mathcal{F}_1$ and therefore one of $L\,\mathrm{Spin}_{2n}\rtimes\mathrm{Rot}\,S^1$ $\diamond$

2.5. The level 1 representations of $L\,\mathrm{Spin}_{2n}$.

We prove below that the level of $\mathcal{F}_R$ and $\mathcal{F}_{NS}$ as positive energy representations of $L\,\mathrm{Spin}_{2n}$ is equal to one by computing the commutator cocycle for the action of the torus and the coroot lattice of Spin_{2n} and using the criterion of corollary I.3.2.4. An alternative infinitesimal proof will be given in section 3. We begin by giving an equivalent description of the Fock spaces $\mathcal{F}_R$ and $\mathcal{F}_{NS}$ parallel to that of the finite–dimensional spin modules obtained in §1.2.

Let $\mathcal{H}=\mathcal{H}_{\mathrm{R}}$ or $\mathcal{H}_{\mathrm{NS}}$ and J the complex structure of §2.4. Split the complexification $\mathcal{H}_{\mathbb{C}}$ as

$$\mathcal{H}_{\mathbb{C}}=\mathcal{H}_{\mathbb{C}}^{1,0}\oplus\mathcal{H}_{\mathbb{C}}^{0,1} \tag{2.5.1}$$

where the summands are the $\pm 1\otimes i$ eigenspaces of $J\otimes 1$. As in the finite–dimensional case, they are isotropic for the $\mathbb{C}$–bilinear extension of the inner product B on $\mathcal{H}$ and are unitarily and anti–unitarily isomorphic to $\mathcal{H}_J$ via the maps

$$U_\pm : f\rightarrow\frac{f\otimes 1-Jf\otimes i}{\sqrt{2}} \tag{2.5.2}$$

If $\mathcal{H}=\mathcal{H}_{\mathrm{NS}}$, $\mathcal{H}_{\mathbb{C}}^{1,0}$ and $\mathcal{H}_{\mathbb{C}}^{0,1}$ are spanned by the functions $v(m)=e^{(1\otimes i)m\theta}\otimes v$ where $v\in V_{\mathbb{C}}$ and $m\in\frac{1}{2}+\mathbb{Z}$ is $\lessgtr 0$ respectively. If, on the other hand $\mathcal{H}=\mathcal{H}_{\mathrm{R}}$, $\mathcal{H}_{\mathbb{C}}^{1,0}$ is spanned by the $v(m)$, with $m\in\mathbb{N}_-$ and $v\in V_{\mathbb{C}}$ or $m=0$ and $v\in V_{\mathbb{C}}^{1,0}$. The latter is a summand of the splitting $V_{\mathbb{C}}=V_{\mathbb{C}}^{1,0}\oplus V_{\mathbb{C}}^{0,1}$ considered in §1.2 and corresponds to the $1\otimes i$–eigenspace of $i\otimes 1$ on $V_{\mathbb{C}}$. Similarly, $\mathcal{H}_{\mathbb{C}}^{0,1}$ is spanned by the $v(m)$ with $m\in\mathbb{N}_+$ and $v\in V_{\mathbb{C}}$ or $m=0$ and $v\in V_{\mathbb{C}}^{0,1}$.

Using U_+, we identify $\mathcal{H}_J$ with $\mathcal{H}_{\mathbb{C}}^{1,0}$ and $\mathcal{F}=\Lambda\mathcal{H}_J$ with $\Lambda\mathcal{H}_{\mathbb{C}}^{1,0}$. Transporting the representation of the Clifford algebra $C(\mathcal{H})$ to the latter Fock space, we find as in §1.2 that it acts as

$$\psi(f)\,g_1\wedge\cdots\wedge g_m=\sqrt{2}\,f\wedge g_1\wedge\cdots\wedge g_m \qquad \text{if } f\in\mathcal{H}_{\mathbb{C}}^{1,0} \tag{2.5.3}$$

and

$$\psi(f)\,g_1\wedge\cdots\wedge g_m=\sum_{j=1}^m(-1)^{j-1}\sqrt{2}\,B(f,g_j)g_1\wedge\cdots\wedge\widehat{g_i}\wedge\cdots\wedge g_m \qquad \text{if } f\in\mathcal{H}_{\mathbb{C}}^{0,1} \tag{2.5.4}$$

where we have extended the definition of the symbols $\psi(\cdot)$ to $\mathbb{C}$–linear symbols $\psi(f)$, $f\in\mathcal{H}_{\mathbb{C}}$ satisfying

$$\{\psi(f),\psi(g)\}=2B(f,g) \qquad \text{and} \qquad \psi(f)^*=\psi(\overline{f}) \tag{2.5.5}$$

We now characterise explicitly the action of the coroot lattice of Spin_{2n} on $\mathcal{F}$. Let $e_1 \cdots e_{2n}$ be an orthonormal basis of V such that $ie_{2k-1} = e_{2k}$. Let T be the torus of SO_{2n} consisting of block diagonal matrices of the form (1.3.6) and $\widetilde{T}$ its pre–image in Spin_{2n}. The coroot lattice $\Lambda_R^\vee \cong \mathrm{Hom}(\mathbb{T}, \widetilde{T}) \subset L\widetilde{T}$ of Spin_{2n} embeds in the integral lattice $I = \mathrm{Hom}(\mathbb{T}, T)$ of SO_{2n}. The latter has a $\mathbb{Z}$–basis given, in the notation of (1.3.7) by $i\Theta_k = ie_{2k} \wedge e_{2k-1}$ where to each $\lambda \in \bigoplus_k i\Theta_k \cdot \mathbb{Z}$, we associate the loop $\zeta_\lambda(\phi) = \exp_T(-i\phi\lambda)$. Let $f_{\pm k} = \sqrt{2}^{-1}(e_{2k-1} \otimes 1 \mp e_{2k} \otimes i)$, $k = 1 \ldots n$ be the orthonormal basis of $V_\mathbb{C}^{1,0}, V_\mathbb{C}^{0,1}$ corresponding to $e_1 \cdots e_{2n}$ via (1.2.1) so that $B(f_k, f_l) = \delta_{k+l,0}$. Then,

LEMMA 2.5.1. *In $\mathcal{F}_R$, we have, projectively,*

$$U(\zeta_{i\Theta_k})\Omega = \frac{1}{\sqrt{2}}\psi(f_{-k}(-1))\Omega = f_{-k}(-1) \tag{2.5.6}$$

PROOF. The proof of proposition 1.2.1 shows that if $R \in O_{\mathrm{res}}(\mathcal{H}, J)$ is such that $R\mathcal{H}_\mathbb{C}^{0,1} \cap \mathcal{H}_\mathbb{C}^{0,1}$ is of finite codimension in $R\mathcal{H}_\mathbb{C}^{0,1}$, then $\Gamma(R)\Omega$ is given, as in (1.2.8) by $\mathrm{Det}(R\mathcal{H}_\mathbb{C}^{0,1}/R\mathcal{H}_\mathbb{C}^{0,1} \cap \mathcal{H}_\mathbb{C}^{0,1})\Omega$. We choose now $R = \zeta_{i\Theta_k}$ and compute $R\mathcal{H}_\mathbb{C}^{1,0}$. From $\Theta_k = e_{2k} \wedge e_{2k-1} = if_k \wedge f_{-k}$, we find $\Theta_k f_{\pm l} = \pm i\delta_{kl} f_{\pm l}$ so that

$$\exp(\sum t_k \Theta_k) f_{\pm l} = e^{\pm i t_l} f_{\pm l} \tag{2.5.7}$$

and therefore

$$\zeta_{i\Theta_k} f_{\pm l}(n)\,(\phi) = e^{(1\otimes i)(n \pm \delta_{kl})\phi} \otimes f_{\pm l} = f_{\pm l}(n \pm \delta_{kl})\,(\phi) \tag{2.5.8}$$

Since

$$\mathcal{H}_\mathbb{C}^{0,1} = \bigoplus_{m>0,l} \mathbb{C}f_l(m) \oplus \mathbb{C}f_{-l}(m) \bigoplus_l \mathbb{C}f_{-l}(0) \tag{2.5.9}$$

we find

$$\zeta_{i\Theta_k}\mathcal{H}_\mathbb{C}^{0,1} = \bigoplus_{m>0,l} \mathbb{C}f_l(m+\delta_{kl}) \oplus \mathbb{C}f_{-l}(m-\delta_{kl}) \bigoplus_l \mathbb{C}f_{-l}(-\delta_{kl}) \tag{2.5.10}$$

so that

$$\zeta_{i\Theta_k}\mathcal{H}_\mathbb{C}^{0,1}/(\zeta_{i\Theta_k}\mathcal{H}_\mathbb{C}^{0,1} \cap \mathcal{H}_\mathbb{C}^{0,1}) \cong \mathbb{C}f_{-k}(-1) \tag{2.5.11}$$

whence the conclusion $\diamond$

LEMMA 2.5.2. *$\mathcal{F}_R$ and $\mathcal{F}_{NS}$ are positive energy representations of $L\,\mathrm{Spin}_{2n}$ of level 1.*

PROOF. By corollary I.3.2.4, it is sufficient to compute the following cocycle on $\Lambda_R^\vee \times \widetilde{T}$ where $\widetilde{T}$ is a maximal torus in Spin_{2n} and $\Lambda_R^\vee \cong \mathrm{Hom}(\mathbb{T}, \widetilde{T})$ the corresponding coroot lattice. Define

$$\kappa(\alpha, \tau) = \Gamma(\zeta_\alpha)\Gamma(\tau)\Gamma(\zeta_\alpha)^*\Gamma(\tau)^* \tag{2.5.12}$$

Since Γ factors through $L\,\mathrm{SO}_{2n}$, κ extends to a bilinear map $I \times T \rightarrow \mathbb{T}$ where $T = \widetilde{T}/\mathbb{Z}_2 \subset \mathrm{SO}_{2n}$ and $I = \mathrm{Hom}(\mathbb{T}, T)$ is the integral lattice spanned by the $i\Theta_k$. Moreover, the inclusion $U_n \subset \mathrm{SO}_{2n}$ identifies the diagonal torus of U_n with T and the corresponding integral lattices. Thus, $I, T \subset LU_n$ and since the Ramond and Neveu–Schwarz Fock spaces are unitarily equivalent as LU_n–modules, we need only compute κ in one of them, say $\mathcal{F}_R$. Notice that $T \subset U_n$ commutes with the complex structure J on $\mathcal{H}_\mathrm{R}$ so that it is canonically quantised and fixes Ω. By lemma 2.5.1 and (2.5.7)

$$\begin{aligned}\Gamma(\zeta_{i\Theta_k})^*\Gamma(\exp_T(h))\Gamma(\zeta_{i\Theta_k})\Gamma(\exp_T(h))^*\Omega &= \Gamma(\zeta_{\Theta_k})^*\Gamma(\exp_T(h))\frac{1}{\sqrt{2}}\psi(f_{-k}(-1))\Omega \\ &= \Gamma(\zeta_{\Theta_k})^*\frac{1}{\sqrt{2}}\psi(\exp_T(h)f_{-k}(-1))\Omega \\ &= e^{\langle i\Theta_k, h\rangle}\Omega\end{aligned} \tag{2.5.13}$$

where $\langle\cdot,\cdot\rangle$ is the basic inner product so that $\langle\Theta_k, \Theta_l\rangle = -\delta_{kl}$. It follows by linearity that

$$\kappa(\alpha, \exp_T(h)) = e^{-\langle\alpha, h\rangle} \tag{2.5.14}$$

and therefore that $\mathcal{F}_R$ and $\mathcal{F}_{NS}$ are of level 1 $\diamond$

Proposition 2.5.3. *Let $\mathcal{H}_0, \mathcal{H}_v, \mathcal{H}_{s_\pm}$ be the level 1 representations of $L\,\mathrm{Spin}_{2n}$ and $\mathcal{F} = \mathcal{F}_{NS}$ or $\mathcal{F}_R$ the Neveu–Schwarz and Ramond Fock spaces. Then, denoting by $\mathcal{F}_0, \mathcal{F}_1$ the even and odd subspaces of $\mathcal{F}$, we have*

$$\mathcal{H}_0 \subseteq \mathcal{F}_{NS,0} \qquad\qquad \mathcal{H}_v \subseteq \mathcal{F}_{NS,1} \tag{2.5.15}$$

and

$$\mathcal{H}_{s_+} \subseteq \mathcal{F}_{R,\epsilon} \qquad\qquad \mathcal{H}_{s_-} \subseteq \mathcal{F}_{R,1-\epsilon} \tag{2.5.16}$$

where $\epsilon = 0$ or 1 according to whether n is even or odd.

Proof. Let $\mathcal{F} = \mathcal{F}_{NS}$. The action of $\mathrm{SO}_{2n} \subset L\,\mathrm{SO}_{2n}$ on the complexification of $\mathcal{H} = \mathcal{H}_{\mathrm{NS}}$ preserves the decomposition (2.5.1) and therefore commutes with the complex structure J. Thus, SO_{2n} is canonically quantised on $\mathcal{F}$ and acts as

$$\Gamma(R)\Omega = \Omega \qquad \text{and} \qquad \Gamma(R)v_1(-n_1) \wedge \cdots \wedge v_k(-n_k) = Rv_1(-n_1) \wedge \cdots \wedge Rv_k(-n_k) \tag{2.5.17}$$

The lowest energy subspace of $\mathcal{F}_0$ is spanned by the vacuum vector Ω and therefore, by (2.5.17), $\mathcal{F}_0 \cong \mathbb{C} \cong V_0$ as SO_{2n}–modules. It follows by lemma I.2.3.2 and lemma 2.4.2 that the closure of the linear span of $\Gamma(L\,\mathrm{Spin}_{2n})\Omega \subset \mathcal{F}_0$ is isomorphic to $\mathcal{H}_0$. Similarly, the lowest energy subspace of $\mathcal{F}_1$ is spanned by the vectors $v(-\frac{1}{2})$, $v \in V_\mathbb{C}$ and by (2.5.17) is isomorphic, as SO_{2n}–module to $V_\mathbb{C}$ so that $\mathcal{H}_v \subseteq \mathcal{F}_1$. Let now $\mathcal{F} = \mathcal{F}_R$. The operators $\psi(v)$, $v \in V_\mathbb{C}$ satisfy $U_\theta \psi(v) U_\theta^* = \psi(R_\theta v) = \psi(v)$ and therefore map $\mathcal{F}(0) = \Lambda V_\mathbb{C}^{1,0}$ to itself. This action coincides in fact with the Clifford algebra multiplication (1.3.9). The projective action of $\mathrm{Spin}_{2n} \subset L\,\mathrm{Spin}_{2n}$ on $\mathcal{F}$ leaves $\mathcal{F}(0)$ invariant and, by (2.3.7) satisfies $\Gamma(S)\psi(v)\Gamma(S)^* = \psi(Sv)$. It follows by the irreducibility of $\Lambda V_\mathbb{C}^{1,0}$ under $C(V)$ that, as Spin_{2n}–modules $\mathcal{F}(0) \cong \Lambda V_\mathbb{C}^{1,0}$. The inclusions (2.5.16) now follow by proposition 1.3.1 and lemma I.2.3.2 $\diamond$

Remark. We shall prove in the next section that the inclusions (2.5.15)–(2.5.16) are in fact equalities by using the infinitesimal action of $L^{\mathrm{pol}}\mathfrak{so}_{2n}$ on $\mathcal{F}_R, \mathcal{F}_{NS}$.

3. The infinitesimal action of $L^{\mathrm{pol}}\mathfrak{so}_{2n}$ on $\mathcal{F}_R, \mathcal{F}_{NS}$

In this section, we prove that the even and odd parts of the Ramond and Neveu–Schwarz Fock spaces are irreducible under the action of $L\,\mathrm{Spin}_{2n}$ and therefore that the inclusions of proposition 2.5.3 are equalities. We proceed by studying the abstract action of $L^{\mathrm{pol}}\mathfrak{so}_{2n}$ on $\mathcal{F}_R$ and $\mathcal{F}_{NS}$ induced by that of $L\,\mathrm{Spin}_{2n}$ and identify it with bilinear expressions in the Fermi field similar to those of §1.3. The irreducibility result then follows from the well–known fact that these give a level 1 representation of the Kac–Moody algebra $\widehat{\mathfrak{so}_{2n}}$ for which the even and odd parts of $\mathcal{F}_R, \mathcal{F}_{NS}$ are irreducible [**IgF, GO2**].

In §3.1, we consider the algebraic Clifford algebra of $\mathcal{H} = \mathcal{H}_{\mathrm{R}}$ or $\mathcal{H}_{\mathrm{NS}}$ generated by the $\psi(f)$ where $f \in \mathcal{H}$ is a trigonometric polynomial and show that it acts irreducibly on the finite energy subspace of the corresponding Fock space. The bilinear expressions giving the action of $\widehat{\mathfrak{so}_{2n}}$ are derived in §3.2. Their identification with the abstract action of $L^{\mathrm{pol}}\mathfrak{so}_{2n}$ then follows by Shur's lemma because both representations have the same commutation relations with the algebraic $\psi(f)$. In §3.3 we derive from this the irreducibility of the even and odd subspaces of $\mathcal{F}_R, \mathcal{F}_{NS}$ under $L\,\mathrm{Spin}_{2n}$.

3.1. Algebraic Fermions.

Let $\mathcal{H} = \mathcal{H}_{\mathrm{R}}$ or $\mathcal{H}_{\mathrm{NS}}$ and $\mathcal{F}$ the corresponding Fock space. The subspace $\mathcal{F}^{\mathrm{fin}} \subset \mathcal{F}$ of finite energy vectors for the canonically quantised action of $\mathrm{Rot}\,S^1$ given by $U_\theta = \Gamma(R_\theta)$ is spanned by the monomials $v_1(-n_1) \wedge \cdots \wedge v_k(-n_k)$ where $n_i \in \mathbb{N}$ for $\mathcal{F}_R$ and $\frac{1}{2} + \mathbb{N}$ for $\mathcal{F}_{NS}$. Consider the *algebraic Clifford algebra* $C^{\mathrm{alg}}(\mathcal{H}) \subset C(\mathcal{H})$ generated by the $\psi(v,n) := \psi(v(n))$, $v \in V_\mathbb{C}$. By (2.3.7)

$$U_\theta \psi(v,n) U_\theta^* = e^{-in\theta}\psi(v,n) \tag{3.1.1}$$

so that $\mathcal{F}_{\mathrm{fin}}$ is invariant under the action of $C^{\mathrm{alg}}(\mathcal{H})$ and, differentiating

$$[d, \psi(v,n)] = -n\psi(v,n) \tag{3.1.2}$$

We prove below that $C^{\mathrm{alg}}(\mathcal{H})$ acts irreducibly on $\mathcal{F}_{\mathrm{fin}}$. The proof is similar to that of lemma 1.1.1 with the replacement of the grading corresponding to the number operator by that given by the infinitesimal generator of rotations d. It relies on the fact that d can be written as a bilinear expression in the $\psi(v,n)$ which we presently derive. To fix notations, if e_i is an orthonormal basis of V, we denote $\psi(e_i,n)$ by $\psi_i(n)$ so that, by (2.5.5)

$$\{\psi_i(n), \psi_j(m)\} = 2\delta_{i,j}\delta_{m+n,0} \qquad \text{and} \qquad \psi_i(n)^* = \psi_i(-n) \tag{3.1.3}$$

Thus, if we consider the $\psi_i(n)$ as anticommuting variables, then $\mathrm{ad}_+(\psi_i(-n)) = 2\dfrac{\partial}{\partial\psi_i(n)}$ where $\mathrm{ad}_+(a)b = \{a,b\}$. Recall the definition of *fermionic normal ordering*

$$\,^\circ_\circ a(n)b(m)^\circ_\circ = \begin{cases} a(n)b(m) & n<m \\ \frac{1}{2}(a(n)b(m) - b(m)a(n)) & m=n \\ -b(m)a(n) & m>n \end{cases} = a(n)b(m) - H(n-m)\{a(n), b(m)\} \tag{3.1.4}$$

where

$$H(p) = \begin{cases} 0 & p<0 \\ \frac{1}{2} & p=0 \\ 1 & p>0 \end{cases} \tag{3.1.5}$$

and the quantum field theoretic prescription that products $\psi_{i_1}(n_1)\cdots\psi_{i_k}(n_k)$ should be written in normal order to prevent the occurrence of infinities. With this in mind, we have using $[ab,c] = a\{b,c\} - \{a,b\}c$

$$\begin{aligned} -4n\psi_i(n) &= \sum_{j,p}\Big(p\psi_j(-p)\{\psi_j(p),\cdot\} - p\{\psi_j(-p),\cdot\}\psi_j(p)\Big)\psi_i(n) \\ &= \sum_{j,p}[p\psi_j(-p)\psi_j(p), \psi_i(n)] \\ &= \sum_{j,p}[p\,^\circ_\circ\psi_j(-p)\psi_j(p)^\circ_\circ, \psi_i(n)] \end{aligned} \tag{3.1.6}$$

where $p \in \mathbb{N}$ for $\mathcal{F}_R$ and $\frac{1}{2}+\mathbb{Z}$ for $\mathcal{F}_{NS}$. The operator $\sum_{j,p} p\,^\circ_\circ\psi_j(-p)\psi_j(p)^\circ_\circ$ is well–defined on $\mathcal{F}^{\mathrm{fin}}$ since the sum is locally finite. In fact,

LEMMA 3.1.1. *The infinitesimal generator of rotations on $\mathcal{F}^{\mathrm{fin}}$ is given by*

$$d = \frac{1}{4}\sum_{j,p} p\,^\circ_\circ\psi_j(-p)\psi_j(p)^\circ_\circ = \frac{1}{2}\sum_{j,p>0} p\psi_j(-p)\psi_j(p) \tag{3.1.7}$$

PROOF. Let D be the operator defined by the right hand–side of (3.1.7). By (3.1.6), $[D,\psi(v,n)] = -n\psi(v,n)$. Moreover, $D\Omega = 0$ where $\Omega \in \mathcal{F}$ is the vacuum vector and therefore by cyclicity of Ω under the action of the $\psi(v,n)$, $D = d$ $\diamond$

PROPOSITION 3.1.2. *$\mathcal{F}^{\mathrm{fin}}$ is an irreducible $C^{\mathrm{alg}}(\mathcal{H})$–module. Moreover, if $T \in \mathrm{End}(\mathcal{F}^{\mathrm{fin}})$ commutes with $C^{\mathrm{alg}}(\mathcal{H})$, then $T = \lambda \cdot 1$ for some $\lambda \in \mathbb{C}$.*

PROOF. Notice first that any vector ξ in $\mathcal{F}^{\mathrm{fin}}(0)$ is cyclic for $C^{\mathrm{alg}}(\mathcal{H})$, *i.e.* $C^{\mathrm{alg}}(\mathcal{H})\xi = \mathcal{F}^{\mathrm{fin}}$. This is obvious if $\mathcal{F} = \mathcal{F}_{NS}$ since $\mathcal{F}^{\mathrm{fin}}(0) = \mathbb{C}\,\Omega$ and follows for $\mathcal{F}_R$ from the irreducibility of $\mathcal{F}^{\mathrm{fin}}(0) = \Lambda V_{\mathbb{C}}^{1,0}$ as $C(V) \subset C^{\mathrm{alg}}(\mathcal{H})$-module which implies $C(V)\xi = \Lambda V_j$. If $T \in \mathrm{End}(\mathcal{F}^{\mathrm{fin}})$ commutes with $C^{\mathrm{alg}}(\mathcal{H})$, it commutes with d by lemma 3.1.1 and therefore leaves $\mathcal{F}^{\mathrm{fin}}(0)$ invariant. Thus, T must have an eigenvector $\xi \in \mathcal{F}^{\mathrm{fin}}(0)$ with eigenvalue λ, and by cyclicity of ξ, $T = \lambda \cdot 1$. If $0 \neq \mathcal{V} \subset \mathcal{F}^{\mathrm{fin}}$ is invariant under $C^{\mathrm{alg}}(\mathcal{H})$, then $d\mathcal{V} \subset \mathcal{V}$ by lemma 3.1.1 so that $\mathcal{V}$ is graded, *i.e.* $\mathcal{V} = \bigoplus_n \mathcal{V}(n)$. The corresponding orthogonal projection $P_{\mathcal{V}} : \mathcal{F}^{\mathrm{fin}} \to \mathcal{V}$ commutes with $C^{\mathrm{alg}}(\mathcal{H})$ and therefore $P_{\mathcal{V}} = 1$ whence $\mathcal{V} = \mathcal{F}^{\mathrm{fin}}$ $\diamond$

3.2. The action of $L^{\mathrm{pol}}\mathfrak{so}_{2n}$ via bilinear expressions in Fermions.

Let $\mathcal{H} = \mathcal{H}_{\mathrm{R}}$ or $\mathcal{H}_{\mathrm{NS}}$ and $\mathcal{F}$ the corresponding Fock space. Denote by $\Gamma : L^{\mathrm{pol}}\mathfrak{so}_{2n} \rightarrow \mathrm{End}(\mathcal{F}^{\mathrm{fin}})$ the projective representation determined by the positive energy action of $L\,\mathrm{Spin}_{2n}$ on $\mathcal{F}$ via theorem I.1.2.1.

LEMMA 3.2.1. *Let $f \in \mathcal{H}_{\mathbb{C}}$ be a trigonometric polynomial so that $\psi(f) \in \mathrm{End}(\mathcal{F}^{\mathrm{fin}})$ and $X \in L^{\mathrm{pol}}\mathfrak{so}_{2n}$. Then*

$$[\Gamma(X), \psi(f)] = \psi(Xf) \tag{3.2.1}$$

PROOF. By theorem I.1.2.1, $\Gamma(X)$ is essentially skew–adjoint on $\mathcal{F}^{\mathrm{fin}}$ and $e^{t\Gamma(X)} = \Gamma(e^{tX})$ in $PU(\mathcal{F})$. By the covariance relation (2.3.7), we have $e^{t\Gamma(X)}\psi(f)e^{-t\Gamma(X)} = \psi(e^{tX}f)$ and applying both sides to $\xi \in \mathcal{F}^{\mathrm{fin}}$, we find $e^{t\Gamma(X)}\psi(f)\xi = \psi(e^{tX}f)e^{t\Gamma(X)}\xi$. The relations (3.2.1) now follow by taking derivatives at $t = 0$. Specifically, $\psi(f)\xi \in \mathcal{F}^{\mathrm{fin}} \subset \mathcal{D}(\Gamma(X))$ and therefore

$$e^{t\Gamma(X)}\psi(f)\xi = \psi(f)\xi + t\Gamma(X)\psi(f)\xi + o(t) \tag{3.2.2}$$

Similarly,

$$\begin{aligned} \psi(e^{tX}f)e^{t\Gamma(X)}\xi &= \psi(f + tXf + o(t))(\xi + t\Gamma(X)\xi + o(t)) \\ &= \psi(f)\xi + t(\psi(Xf)\xi + \psi(f)\Gamma(X)\xi) + o(t) \end{aligned} \tag{3.2.3}$$

since, by (2.5.5), $\|\psi(g)\| \leq \sqrt{2}\|g\|$. Letting $t \rightarrow 0$ yields (3.2.1) $\diamond$

Specialising the above, we find

$$[E_{ij}(n), \psi_k(m)] = \delta_{j,k}\psi_i(m+n) - \delta_{i,k}\psi_j(m+n) \tag{3.2.4}$$

where we denote $\Gamma(X)$ and X by the same symbol and $E_{ij} \in \mathfrak{so}_{2n} \cong \Lambda^2 V$ is the basis given by $e_i \wedge e_j$. Because of the irreducibility of the action of $C^{\mathrm{alg}}(\mathcal{H})$, the above equations have essentially unique solutions in the $E_{ij}(n)$. To solve them, notice that the right hand–side of (3.2.4) may be rewritten using the identity $a\{b,c\} - \{a,c\}b = [ab, c]$ as

$$\begin{aligned} &\frac{1}{2}\sum_p \Big(\psi_i(p+n)\{\psi_j(-p), \cdot\} - \{\psi_i(-p), \cdot\}\psi_j(p+n)\Big)\psi_k(m) = \\ &\frac{1}{2}\sum_p \Big(\psi_i(p+n)\{\psi_j(-p), \cdot\} - \{\psi_i(p+n), \cdot\}\psi_j(-p)\Big)\psi_k(m) = \\ &\frac{1}{2}\sum_p [\psi_i(p+n)\psi_j(-p), \psi_k(m)] = \\ &\frac{1}{2}\sum_p [{}^\circ_\circ\psi_i(p+n)\psi_j(-p){}^\circ_\circ, \psi_k(m)] \end{aligned} \tag{3.2.5}$$

Define now

$$E_{ij}(n) = \frac{1}{2}\sum_p {}^\circ_\circ\psi_i(n+p)\psi_j(-p){}^\circ_\circ \tag{3.2.6}$$

and use the above to define operators $X(n)$ for any $X \in \mathfrak{so}_{2n}$. The following result is well–known, see for example [**IgF**]

PROPOSITION 3.2.2. *For $X, Y \in \mathfrak{so}_{2n}$, the operators $X(n)$ satisfy*

(i) $[X(n), \psi(v,m)] = \psi(Xv, m+n)$
(ii) $[X(n), Y(m)] = [X,Y](n+m) + n\delta_{n+m,0}\langle X, Y\rangle$

where $\langle\cdot,\cdot\rangle$ is the basic inner product on $\mathfrak{so}_{2n}$.

PROOF. (i) follows from (3.2.5). From (i) we deduce that $[X(n), Y(m)] - [X,Y](n+m)$ commutes with the $\psi(v,n)$ and hence by proposition 3.1.2 is equal to a scalar $\lambda(X(n), Y(m))$. Since $[d, \psi(v,n)] = -n\psi(v,n)$, the operators $Z(p)$ are homogeneous of degree $-p$ and therefore λ vanishes unless $m+n = 0$.

We may compute it in that case by evaluating $(\Omega, [X(n), Y(-n)]\Omega) - (\Omega, [X, Y](0)\Omega)$ and choosing for X, Y basis elements since λ is bilinear in its arguments. From

$$[ab, cd] = \{b, c\}ad - \{a, d\}cb + \{b, d\}ca - \{a, c\}bd \tag{3.2.7}$$

which holds if all anti–commutators are scalars, we find for $n \geq 0$

$$\begin{aligned}(\Omega, [E_{ij}(n), E_{kl}(-n)]\Omega) &= \sum_{0\leq p,q\leq n} (\Omega, [\frac{1}{2}\psi_i(n-p)\psi_j(p), \frac{1}{2}\psi_k(-n+q)\psi_l(-q)]\Omega) \\ &= \sum_{0\leq p,q\leq n} \delta_{j,k}\delta_{p,n-q}(\Omega, \frac{1}{2}\psi_i(q)\psi_l(-q)\Omega) - \delta_{i,l}\delta_{q,n-p}(\Omega, \frac{1}{2}\psi_k(-p)\psi_j(p)\Omega) \\ &+ \sum_{0\leq p,q\leq n} \delta_{j,l}\delta_{p,q}(\Omega, \frac{1}{2}\psi_k(-p)\psi_i(p)\Omega) - \delta_{i,k}\delta_{p,q}(\Omega, \frac{1}{2}\psi_j(p)\psi_l(-p)\Omega)\end{aligned} \tag{3.2.8}$$

where $p, q \in \mathbb{Z}$ for $\mathcal{F}_R$ and $\frac{1}{2} + \mathbb{Z}$ for $\mathcal{F}_{NS}$. Using (3.1.4), we may rewrite (3.2.8) as

$$\begin{aligned}&\sum_{0\leq p\leq n} \delta_{j,k}(\Omega, \frac{1}{2}{:}\psi_i(p)\psi_l(-p){:}\Omega) + \delta_{i,l}(\Omega, \frac{1}{2}{:}\psi_j(p)\psi_k(-p){:}\Omega) \\ -&\sum_{0\leq p\leq n} \delta_{j,l}(\Omega, \frac{1}{2}{:}\psi_i(p)\psi_k(-p){:}\Omega) + \delta_{i,k}(\Omega, \frac{1}{2}{:}\psi_j(p)\psi_l(-p){:}\Omega) \\ +&\sum_{0\leq p\leq n} \delta_{i,l}\delta_{j,k}\Big(H(2p) - H(-2p)\Big) - \delta_{i,k}\delta_{j,l}\Big(H(2p) - H(-2p)\Big) \\ =&\delta_{j,k}(\Omega, E_{il}(0)\Omega) + \delta_{i,l}(\Omega, E_{jk}(0)\Omega) - \delta_{j,l}(\Omega, E_{ik}(0)\Omega) - \delta_{i,k}(\Omega, E_{jl}(0)\Omega) + n(\delta_{i,l}\delta_{j,k} - \delta_{i,k}\delta_{j,l}) \\ =&(\Omega, [E_{ij}, E_{kl}](0)\Omega) + n(\delta_{i,l}\delta_{j,k} - \delta_{i,k}\delta_{j,l})\end{aligned} \tag{3.2.9}$$

The following lemma then shows that (ii) holds $\diamond$

LEMMA 3.2.3. *Let $\langle\cdot,\cdot\rangle$ be the basic inner product on $\mathfrak{so}_{2n}$. Then, $\langle E_{ij}, E_{kl}\rangle = \delta_{j,k}\delta_{i,l} - \delta_{j,l}\delta_{i,k}$.*

PROOF. Let $V = \mathbb{R}^{2n}$. The form $\tau(X, Y) = \mathrm{tr}_V(XY)$ is symmetric, bilinear and ad–invariant and is therefore a multiple β of $\langle\cdot,\cdot\rangle$. To compute β, let h_i and h^i be dual basis of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{so}_{2n}$ with respect to $\langle\cdot,\cdot\rangle$. Denoting by $\Pi(V_{\mathbb{C}})$ the weights of $V_{\mathbb{C}}$, and computing with a basis of $V_{\mathbb{C}}$ diagonal for the action of t, we get

$$\beta \dim \mathfrak{t} = \tau(h_i, h^i) = \sum_{\mu\in\Pi(V)} \mu(h_i)\mu(h^i) = \sum_{\mu\in\Pi(V)} \|\mu\|^2 \tag{3.2.10}$$

Since the weights of $V_{\mathbb{C}}$ are $\pm\theta_i$, we get $\beta = 2$. To conclude, notice that in V, $E_{ij}E_{kl} = \delta_{jk}\varepsilon_{il} + \delta_{il}\varepsilon_{jk} - \delta_{ik}\varepsilon_{jl} - \delta_{jl}\varepsilon_{ik}$ where $\epsilon_{pq}e_r = \delta_{qr}e_p$ and therefore $\mathrm{tr}_V(E_{ij}E_{kl}) = 2(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})$ $\diamond$

3.3. Finite reducibility of $\mathcal{F}_R, \mathcal{F}_{NS}$ under $L\,\mathrm{Spin}_{2n}$.

PROPOSITION 3.3.1. *Let $\mathcal{F}$ be the Neveu–Schwarz or Ramond sector. Then, the even and odd subspaces of $\mathcal{F}$ are irreducible under $L\,\mathrm{Spin}_{2n}$ and therefore*

$$\mathcal{F}_{NS} = \mathcal{H}_0 \oplus \mathcal{H}_v \qquad \textit{and} \qquad \mathcal{F}_R = \mathcal{H}_{s_+} \oplus \mathcal{H}_{s_-} \tag{3.3.1}$$

PROOF. Let Γ and π be the projective representations of $L^{\mathrm{pol}}\mathfrak{so}_{2n}$ on $\mathcal{F}^{\mathrm{fin}}$ determined by the action of $L\,\mathrm{Spin}_{2n}$ and the bilinears (3.2.6) respectively. By lemma 3.2.1 and proposition 3.2.2, Γ and π have the same commutation relations with the $\psi(f)$, f a trigonometric polynomial and it follows by proposition 3.1.2 that $\Gamma(X) = \pi(X) + \alpha(X)$ for some $\alpha(X) \in \mathbb{C}$. Since the $\pi(X)$ act irreducibly on the even

and odd subspaces of $\mathcal{F}^{\rm fin}$ [**IgF**, thm. I.3.21], [**GO2**, §5.6] so do the $\Gamma(X)$ and therefore, by proposition I.1.2.2 $\mathcal{F}_0, \mathcal{F}_1$ are irreducible under $L\,\mathrm{Spin}_{2n}$. The conclusion now follows from proposition 2.5.3 ◇

REMARK. The fact that the abstract action of $L^{\rm pol}\mathfrak{so}_{2n}$ and that given by the bilinears (3.2.6) coincide gives, by proposition 3.2.2 another way of showing that $\mathcal{F}$ is a level 1 representation of $L\,\mathrm{Spin}_{2n}$.

4. The level 1 vector primary fields

We show below that the level 1 vector primary fields define bounded operator–valued distributions by identifying them with the Fermi fields within the construction of the level 1 representations of $L\,\mathrm{Spin}_{2n}$ given in sections 2–3. The same result holds in fact at any level and will be established in this generality in chapter VI.

Let $V_{\mathbb{C}}$ be the complexified defining representation of SO_{2n} and V_0, $V_{s_\pm}$ the trivial and spin modules. By the tensor product rules of proposition I.2.2.2, the spaces $\mathrm{Hom}_{\mathrm{Spin}_{2n}}(V_i\otimes V_{\mathbb{C}}, V_j)$ where V_i, V_j are admissible at level 1 are non–zero if, and only if $\{V_i,V_j\}=\{V_0,V_{\mathbb{C}}\}$ or $\{V_i,V_j\}=\{V_{s_+},V_{s_-}\}$. Thus, by proposition 3.3.1, the non–zero vector primary fields at level 1 are endomorphisms of the spaces of finite energy vectors of the Ramond and Neveu–Schwarz Fock spaces. We shall construct them as such.

PROPOSITION 4.1. *Let $\phi:\mathcal{H}_i^{\rm fin}\otimes V_{\mathbb{C}}[z,z^{-1}]\rightarrow\mathcal{H}_j^{\rm fin}$ be a vector primary field at level 1. Then, ϕ extends to a bounded map $L^2(S^1,V_{\mathbb{C}})\rightarrow\mathcal{B}(\mathcal{H}_i,\mathcal{H}_j)$ intertwining the action of $L\,\mathrm{Spin}_{2n}\rtimes\mathrm{Rot}\,S^1$ and satisfying*

$$\|\phi(f)\|\leq\sqrt{2}\|f\| \tag{4.1}$$

PROOF. We begin with the Ramond sector $\mathcal{H}=\mathcal{H}_{\rm R}$. Since $\mathcal{H}_{\mathbb{C}}=L^2(S^1,V_{\mathbb{C}})$ we may define a map $\phi:L^2(S^1,V_{\mathbb{C}})\rightarrow\mathcal{B}(\mathcal{F}_R)$ by $f\rightarrow\psi(f)$ where the latter acts by (2.5.3)–(2.5.4). If P_ϵ, $\epsilon=0,1$ are the orthogonal projections onto the even and odd subspaces of $\mathcal{F}_R$, we claim that the restrictions of $P_\epsilon\phi(\cdot)P_{1-\epsilon}$ to $V_{\mathbb{C}}[z,z^{-1}]$ are primary fields of charge $V_{\mathbb{C}}$. We have already noted in §3.1 that if f is a polynomial, $\psi(f)$ defines an endomorphism of the finite energy subspace $\mathcal{F}^{\rm fin}$. Moreover, by lemma 3.2.1 and the fact that $\mathcal{F}_0,\mathcal{F}_1$ are invariant under $L\,\mathrm{Spin}_{2n}$, $P_\epsilon\phi(\cdot)P_{1-\epsilon}$ intertwines the action of $L^{\rm pol}\mathfrak{so}_{2n}$ as claimed. Notice in passing that the initial terms of $P_\epsilon\phi(\cdot)P_{1-\epsilon}$, that is the restriction to Spin_{2n} intertwiners

$$\Lambda_{1-\epsilon}V_{\mathbb{C}}^{1,0}=\mathcal{F}_{1-\epsilon}(0)\rightarrow\mathcal{F}_\epsilon(0)=\Lambda_\epsilon V_{\mathbb{C}}^{1,0} \tag{4.2}$$

coincided with the Clifford multiplication map given by (1.3.9). By construction, the $P_\epsilon\phi(\cdot)P_{1-\epsilon}$ extend to $L^2(S^1,V_{\mathbb{C}})$ and, by (2.3.7) intertwine $L\,\mathrm{Spin}_{2n}\rtimes\mathrm{Rot}\,S^1$. Moreover, by (2.5.5)

$$\|\psi(f)\omega\|^2+\|\psi(f)^*\omega\|=2\|f\|^2\|\omega\|^2 \tag{4.3}$$

and (4.1) follows. The construction of the vector primary fields in the Neveu–Schwarz sector $\mathcal{H}=\mathcal{H}_{\rm NS}$ proceeds in exactly the same way. Consider unitaries $S_\pm:L^2(S^1,V_{\mathbb{C}})\rightarrow\mathcal{H}_{\mathbb{C}}$ given by $f\rightarrow e^{\pm\frac{1}{2}(1\otimes i)\theta}f$. These intertwine the actions of $L\,\mathrm{Spin}_{2n}$ and satisfy $S^\pm R_\theta S^{\pm^*}=e^{\pm\frac{1}{2}(1\otimes i)\theta}R_\theta$. If P_0, P_1 are the projections onto the even and odd subspaces of $\mathcal{F}_{NS}$, then $f\rightarrow P_1\psi(S_-f)P_0, P_0\psi(S_+f)P_1$ are easily seen to be the required primary fields ◇

CHAPTER IV

Local loop groups and their associated von Neumann algebras

This chapter assembles various results required for the definition of Connes fusion to be given in chapter IX.

The notion of *Connes fusion* of positive energy representations of LG, $G=\mathrm{Spin}_{2n}$ arises by regarding them as bimodules over the subgroups $L_IG, L_{I^c}G$ of loops supported in a given interval $I\subset S^1$ and its complement and using a tensor product operation on bimodules over von Neumann algebras due to Connes [**Co, Sa**]. Recall that a bimodule $\mathcal{H}$ over a pair (M,N) of von Neumann algebras is a Hilbert space supporting commuting representations of M and N. To any two bimodules X,Y over the pairs (M,N), $(\widetilde{N},P)$, Connes fusion associates an (M,P)–bimodule denoted by $X\boxtimes Y$. The definition of $X\boxtimes Y$ relies on, but is ultimately independent of the choice of a reference or *vacuum* $(N,\widetilde{N})$–bimodule $\mathcal{V}$ with a cyclic vector Ω for both actions and for which *Haag duality* holds, *i.e.* the actions of N and $\widetilde{N}$ are each other's commutant. Given $\mathcal{V}$, we form the intertwiner spaces $\mathfrak{X}=\mathrm{Hom}_N(\mathcal{V},X)$ and $\mathfrak{Y}=\mathrm{Hom}_{\widetilde{N}}(\mathcal{V},Y)$ and consider the sequilinear form on the algebraic tensor product $\mathfrak{X}\otimes\mathfrak{Y}$ given by

$$\langle x_1\otimes y_1, x_2\otimes y_2\rangle = (x_2^*x_1y_2^*y_1\Omega,\Omega)$$

where the inner product on the right hand–side is taken in $\mathcal{V}$. If $x_1=x_2$ and $y_1=y_2$, Haag duality implies that $x_2^*x_1$ and $y_2^*y_1$ are commuting positive operators and therefore that $\langle\cdot,\cdot\rangle$ is positive semi–definite. By definition, the bimodule $X\boxtimes Y$ is the Hilbert space completion of $\mathfrak{X}\otimes\mathfrak{Y}$ with respect to $\langle\cdot,\cdot\rangle$, with (M,P) acting as $(m,p)x\otimes y = mx\otimes py$.

Applying the above to the positive energy representations of LG requires a number of preliminary results which are established in this chapter. Let $\mathcal{P}_\ell$ be the set of positive energy representations at a fixed level ℓ. We wish to regard any $(\mathcal{H},\pi)\in\mathcal{P}_\ell$ as a bimodule over the pair $\pi_0(L_IG)^{''},\pi_0(L_{I^c}G)^{''}$ where π_0 is the vacuum representation at level ℓ whose lowest energy subspace is, by definition the trivial G–module. The well–foundedness of this change of perspective is justified by the following properties

(i) *Locality* : $\pi(L_IG)''\subset\pi(L_{I^c}G)'$ for any $(\pi,\mathcal{H})\in\mathcal{P}_\ell$. In other words, $\mathcal{H}$ is a $(\pi(L_IG)^{''},\pi(L_{I^c}G)^{''})$ bimodule.

(ii) *Local equivalence* : All $(\pi,\mathcal{H})\in\mathcal{P}_\ell$ are unitarily equivalent as L_IG–modules. Thus we may unambiguously identify $\pi(L_IG)''$ with $\pi_0(L_IG)''$ where $\pi_0\in\mathcal{P}_\ell$ is the vacuum representation and consider $\mathcal{H}$ as a $(\pi_0(L_IG)'',\pi_0(L_{I^c}G)'')$–bimodule.

(iii) *von Neumann density* : $\pi(L_IG)\times\pi(L_{I^c}G)$ is strongly dense in $\pi(LG)$. Thus, inequivalent irreducible positive energy representations of LG remain so when regarded as bimodules.

The rôle of the reference bimodule is played by the vacuum representation $(\pi_0,\mathcal{H}_0)$. Two crucial facts need to be established in this respect

(iv) *Reeh-Schlieder theorem* : any finite energy vector of a positive energy representation π is cyclic under $\pi(L_IG)$. In particular, the lowest energy vector $\Omega\in\mathcal{H}_0(0)$ is cyclic for $\pi_0(L_IG)''$ and $\pi_0(L_{I^c}G)''$.

(v) *Haag duality* : $\pi_0(L_IG)^{''}=\pi_0(L_{I^c}G)^{'}$.

Finally, another technically crucial property of the algebras $\pi(L_IG)''$ is the following

(vi) *Factoriality* : The algebras $\pi(L_IG)''$ with $I\subsetneq S^1$ are type III_1 factors.

Properties (i)-(vi) were established by Jones and Wassermann for $G = SU_n$ in [**Wa3**] and, in unpublished notes for all simple, simply–connected compact Lie groups using the quark model and conformal inclusions [**Wa1**]. In section 1, we give a model independent proof of locality and von Neumman density in the latter generality. The remaining properties are proved in section 2 by following Jones and Wassermann's original lines.

1. Local loop groups

For any open, possibly improper, interval $I \subseteq S^1$, define the local loop group $L_IG = \{f \in LG |\ f|_{S^1 \setminus I} \equiv 1\}$. The complementary interval I^c is, by definition $S^1 \setminus \overline{I}$.

1.1. Locality.

The following is an immediate generalisation of proposition 3.4.1. of [**PS**].

Lemma 1.1.1. *Assume G is simple. Then for any open interval $I \subseteq S^1$, L_IG is perfect i.e. it is equal to its commutator subgroup. In particular,* $\mathrm{Hom}(L_IG, \mathbb{T}) = \{1\}$.

Proof. Consider the commutator map $C : G \times G \to G$, $(g, h) \to ghg^{-1}h^{-1}$. Its differential at $(1, 1)$ is given by the bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and is therefore surjective since G is simple. By the implit function theorem, C has a smooth right inverse defined on a neighborhood U of the identity and mapping 1 to $(1, 1)$. It follows that $L_IU \subset L_IG$ is contained in $[L_IG, L_IG]$ and therefore $L_IG = \bigcup_n L_IU^n \subset [L_IG, L_IG]$ since L_IG is connected $\diamond$

Proposition 1.1.2 (Locality). *If $I, J \subset S^1$ are disjoint open intervals and $(\pi, \mathcal{H})$ is a positive energy representation of LG, then*

$$\pi(L_IG)'' \subset \pi(L_JG)' \tag{1.1.1}$$

Proof. For any $\gamma_I, \gamma_J \in L_IG \times L_JG$ the following holds in $U(\mathcal{H})$: $\pi(\gamma_I)\pi(\gamma_J)\pi(\gamma_I)^*\pi(\gamma_J)^* = \chi(\gamma_I, \gamma_J)$, where $\chi(\gamma_I, \gamma_J) \in \mathbb{T}$ does not depend upon the choices of the particular lifts and is multiplicative in either variable. By lemma 1.1.1, $\chi \equiv 1$ and $\pi(L_IG)$ commutes with $\pi(L_JG)$ $\diamond$

1.2. The Sobolev $\frac{1}{2}$ space.

We establish below the density of the space of smooth, complex–valued functions vanishing to all orders on a finite subset of S^1 in $C^\infty(S^1)$ for the Sobolev $\frac{1}{2}$–norm. We shall use this result in §1.3 to prove von Neumann density by showing that the topology on LG obtained by pulling back the strong operator topology on the projective unitary group of a positive energy representation is weaker than the Sobolev $\frac{1}{2}$–topology. The discussion below is taken from [**Wa1**, Chap. VIII].

Let $A \subset S^1$ be a finite subset and denote by $C^k_A(S^1)$, $k \in \mathbb{N} \cup \{\infty\}$, the space of smooth functions vanishing up to order k on A. If $f : S^1 \to \mathbb{C}$ is a trigonometric polynomial with Fourier series $\sum_k a_k e^{ik\theta}$ and $s \in \mathbb{R}$, define $\|f\|_s^2 = \sum_k (1+|k|)^{2s}|a_k|^2$ and $|f|_s = \sum_k (1+|k|)^s |a_k|$. Denote by $H^s(S^1)$ the completion of the space of trigonometric polynomials with respect to the norm $\| \cdot \|_s$. It contains $C^\infty(S^1)$ since the Fourier coefficents of a smooth function decrease faster than any polynomial.

Lemma 1.2.1. *If $f, g \in C^\infty(S^1)$ and $s \geq 0$, then $\|fg\|_s \leq |f|_s \|g\|_s$.*

Proof. Consider first the case $f = a_k e^{ik\theta}$ and let $g = \sum_n b_n e^{in\theta}$. Then

$$\|fg\|_s^2 = |a_k|^2 \sum_n (1 + |n|)^{2s} |b_{n-k}|^2 \leq |a_k|^2 (1 + |k|)^{2s} \sum_n (1 + |n - k|)^{2s} |b_{n-k}|^2 = |a_k|^2 (1 + |k|)^{2s} \|g\|_s^2 \tag{1.2.1}$$

since $(1+|p+q|) \leq (1+|p|)(1+|q|)$. Thus, if $f = \sum_k a_k e^{ik\theta}$, we have $\|fg\|_s \leq \sum_k |a_k|(1+|k|)^s \|g\|_s = |f|_s \|g\|_s$ $\diamond$

Lemma 1.2.2. *$C^1_A(S^1)$ is dense in $C^\infty(S^1)$ for the $\| \cdot \|_{\frac{1}{2}}$ norm.*

PROOF. Consider first the case $A = \{0\}$. Let $f_n = \sum_{k=2}^{n} \frac{\cos(k\theta)}{k \log k}$ and set $\chi_n = f_n f_n(0)^{-1}$. Then $f = \lim_n f_n$ exists in $H^{\frac{1}{2}}$ but $\lim_n f_n(0) = \infty$ and therefore $\chi_n(0) = 1$, $\chi_n'(0) = 0$ and $\chi_n \to 0$ in $H^{\frac{1}{2}}$. Thus, if $h \in C^\infty(S^1)$, then $h_n = h(1-\chi_n) \in C^1_A(S^1)$ and by the above lemma, $\|h-h_n\|_{\frac{1}{2}} \leq |h|_{\frac{1}{2}} \|\chi_n\|_{\frac{1}{2}}$ which tends to zero. The general case $A = \{\theta_1, \ldots, \theta_k\}$ follows easily ◇

PROPOSITION 1.2.3. *$C^\infty_A(S^1)$ is dense in $C^\infty(S^1)$ for the $\|\cdot\|_{\frac{1}{2}}$ norm.*

PROOF. It is sufficient to establish the density of $C^\infty_A(S^1)$ in $C^1_A(S^1)$. Moreover, since

$$\|f\|^2_{\frac{1}{2}} \leq \sum_k |a_k|^2(1+k^2) = \|f\|^2_{L^2} + \|\dot{f}\|^2_{L^2} \leq \|f\|^2_\infty + \|\dot{f}\|^2_\infty \leq (4\pi^2+1)\|\dot{f}\|^2_\infty \tag{1.2.2}$$

for $f \in C^1_A(S^1)$, density for the C^1 norm will do. Let $f \in C^1_A(S^1)$. By the Stone-Weierstrass theorem, there exists a sequence $g_n \in C^\infty_c(S^1 \backslash A)$ such that $\|\dot{f} - g_n\|_\infty \to 0$. If $A = \{\theta_1, \ldots, \theta_n\}$ with the points arranged in incresing order, choose ρ_i smooth and supported in (θ_i, θ_{i+1}) with $\int \rho_i = 1$. Set $g_n' = g_n - \sum_i \rho_i \int_{\theta_i}^{\theta_{i+1}} g_n(t)dt$ and $G_n(\theta) = \int_{\theta_1}^{\theta} g_n'(t)dt$ so that $G_n \in C^\infty_c(S^1\backslash A)$. Since $\int_{\theta_i}^{\theta_{i+1}} \dot{f} = 0$, we have

$$\|\dot{f} - \dot{G}_n\|_\infty \leq \|\dot{f} - g_n\|_\infty + \sum_i \|\rho_i\|_\infty \int_{\theta_i}^{\theta_{i+1}} |g_n - \dot{f}| dt \to 0 \tag{1.2.3}$$

◇

1.3. Von Neumann density.

Our proof of von Neumann density follows Jones and Wassermann's original lines [**Wa1**]. It bypasses however the use of conformal inclusions to obtain the general case from SU_n or Spin_{2n} by making use of the Sobolev estimates of Goodman and Wallach [**GoWa**, §3.2].

Let $I \subseteq S^1$ be an open interval and $A \subset I$ a finite set. Denote by $L^A_I G \subset L_I G$ the normal subgroup of loops equal to 1 to all orders on A. Clearly, if $I\backslash A = I_1 \sqcup \ldots \sqcup I_k$, then $L^A_I G = L_{I_1}G \times \cdots \times L_{I_k}G$. We will prove that $\pi(L^A_I G)$ is dense in $\pi(L_I G)$ for any positive energy representation $(\pi, \mathcal{H})$. We shall need for this purpose a different version of the Sobolev estimates governing the action of $L\mathfrak{g}_{\mathbb{C}}$ on the space of smooth vectors $\mathcal{H}^\infty$ obtained in proposition II.1.2.1. For $X = \sum_k X(k) \in L\mathfrak{g}_{\mathbb{C}}$ and $s \in \mathbb{R}$ define $\|X\|^2_s = \sum_k (1+|k|)^{2s}|X(k)|^2$ and recall that $|X|_s = \sum_k (1+|k|)^s |X(k)|$ where $|\cdot|$ is a norm on $\mathfrak{g}_{\mathbb{C}}$. Denote as customary by L_0 the infinitesimal generator of rotations given by the Segal–Sugawara formula (II.1.2.1).

LEMMA 1.3.1. *For any $X \in L\mathfrak{g}_{\mathbb{C}}$, $\xi \in \mathcal{H}^\infty$ and $\epsilon > 0$ the following holds*

$$\|\pi(X)\xi\| \leq C_\epsilon \|X\|_{\frac{1}{2}} \|(1+L_0)^{1+\epsilon}\xi\| \tag{1.3.1}$$

PROOF. Let $X = X(k) \in \mathfrak{g}_{\mathbb{C}} \otimes e^{ik\theta}$ and $\xi \in \mathcal{H}$ be a finite energy vector. By the estimates of proposition II.1.2.1,

$$\|\pi(X)\xi\|^2 \leq C^2(1+|k|)|X(k)|^2 \|(1+L_0)^{\frac{1}{2}}\xi\|^2 \tag{1.3.2}$$

Thus, if $X = \sum_k X(k)$ and ξ is an eigenvector of L_0 so that the $\pi(X(k))\xi$ are orthogonal, we get $\|\pi(X)\xi\| \leq C\|X\|_{\frac{1}{2}} \|(1+L_0)^{\frac{1}{2}}\xi\|$. More generally, if $\xi = \sum_m \xi_m$ with $L_0\xi_m = m\xi_m$ and $\epsilon > 0$, then

$$\begin{aligned}\|\pi(X)\xi\| &\leq C\|X\|_{\frac{1}{2}} \sum_m \|(1+L_0)^{\frac{1}{2}}\xi_m\| \\ &\leq C\|X\|_{\frac{1}{2}} \Big(\sum_m (1+m)^{-1-2\epsilon}\Big)^{\frac{1}{2}} \Big(\sum_m \|(1+L_0)^{1+\epsilon}\xi_m\|^2\Big)^{\frac{1}{2}} \\ &\leq C_\epsilon \|X\|_{\frac{1}{2}} \|(1+L_0)^{1+\epsilon}\xi\|\end{aligned} \tag{1.3.3}$$

◇

PROPOSITION 1.3.2. *Let π be a positive energy representation of LG. Then $\pi(L^A_I G)$ is strongly dense in $\pi(L_I G)$.*

PROOF. Since L_IG is connected, is suffices to show that $\pi(V) \subset \overline{\pi(L_I^A G)}$ for a suitable neighborhhod $1 \in V \subset L_IG$ for then $\pi(L_IG) = \bigcup_n \pi(V)^n \subset \overline{\pi(L_I^A G)}$. Let $\mathcal{U}$ be a neighborhood of the identity in G on which the logarithm is well-defined and set $V = L_I\mathcal{U}$. If $\gamma = \exp_{LG}(X) \in L_IG$, we claim that for any $\xi \in \mathcal{H}$, $\pi(\gamma)\xi = \exp(\pi(X))\xi$ may be arbitrarily well approximated by $\exp(\pi(Y))\xi$ with $Y \in L_I^A\mathfrak{g}$. By the norm-boundedness of the unitaries $\pi(\gamma)$ it is sufficent to prove this on the dense subspace of vectors $\xi \in \mathcal{H}^\infty \subset \mathcal{H}$. Since these are invariant under LG by proposition II.1.5.3, the function $F(t) = e^{-t\pi(Y)}e^{t\pi(X)}\xi$ is smooth and $\dot{F}(t) = e^{-t\pi(Y)}(\pi(X) - \pi(Y))e^{t\pi(X)}\xi$. Therefore,

$$\begin{aligned} \|e^{\pi(Y)}\xi - e^{\pi(X)}\xi\| &= \|e^{\pi(Y)}(1 - e^{-\pi(Y)}e^{\pi(X)})\xi\| \\ &\leq \int_0^1 \|e^{\pi(Y)}\dot{F}(t)\| dt \\ &\leq C_\epsilon M_X \|X - Y\|_{\frac{1}{2}} \|(1 + L_0)^{1+\epsilon}\xi\| \end{aligned} \tag{1.3.4}$$

where we have used that LG acts continuously on each Sobolev space $\mathcal{H}^s$. The result now follows from the density of $L_I^A\mathfrak{g}$ in $L_I\mathfrak{g}$ with respect to the $\|\cdot\|_{\frac{1}{2}}$ norm given by proposition 1.2.3 ◇

COROLLARY 1.3.3 (von Neumann density). *On any positive energy representation π, we have*

$$\pi(L_IG)'' \vee \pi(L_{I^c}G)'' = \pi(LG)'' \tag{1.3.5}$$

REMARK. The Sobolev estimates of proposition II.1.2.1, namely $\|\pi(X)\xi\| \leq C|X|_{\frac{1}{2}}\|(1+L_0)^{\frac{1}{2}}\xi\|$ where $|X|_s = \sum_k |X(k)|(1 + |k|)^s$ cannot be used directly to establish the above density result. Indeed, $|\cdot|_s$ dominates the $C(S^1, \mathfrak{g})$ norm for any $s \geq 0$ and therefore $L_I^A\mathfrak{g}$ is **not** dense in $L_I\mathfrak{g}$ for the corresponding topology.

1.4. The Reeh–Schlieder theorem.

PROPOSITION 1.4.1 (Reeh–Schlieder theorem). *Let $(\pi, \mathcal{H})$ be an irreducible positive energy representation of LG and $I \subset S^1$ an open interval. Then, any finite energy vector $\xi \in \mathcal{H}$ is cyclic for L_IG i.e. the linear span of $\pi(L_IG)\xi$ is dense in $\mathcal{H}$.*

PROOF. This is proved in [**Wa3**, §17] ◇

COROLLARY 1.4.2. *If $I \subset S^1$ is an open non-dense interval and $(\pi, \mathcal{H})$ an irreducible positive energy representation of LG then any finite energy-vector $\xi \in \mathcal{H}$ is cyclic and separating for $M_I = \pi(L_IG)''$, i.e. $\overline{M_I\xi} = \overline{M_I'\xi} = \mathcal{H}$.*

PROOF. The result follows at once from the Reeh–Schlieder theorem and locality since $M_I' \supset \pi(L_{I^c}G)$ ◇

2. Von Neumann algebras generated by local loops groups

In this section, we establish the Haag duality, local equivalence and factoriality properties for the von Neumann algebras generated by local loop groups L_IG in positive energy representations.

The proof of Haag duality relies on the modular theory of Tomita and Takesaki which we outline in §2.1. This allows in principle to compute the commutant of a von Neumann algebra M acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ω, in that it yields the existence of a canonical conjugation J on $\mathcal{H}$ such that $M' = JMJ$. When M is the von Neumann algebra generated by a local loop group L_IG in the vacuum representation $\mathcal{H}_0$ and I is the upper semi–circle, the action of J corresponds to the map $z \to z^{-1}$ on S^1 exchanging I and I^c. It follows from this that $\pi_0(L_IG)' = \pi_0(L_{I^c}G)''$ and therefore that Haag duality holds. This explicit characterisation of J is obtained in §2.3 by using the fermionic realisation of $\mathcal{H}_0$ obtained in chapter III and a computation of the modular conjugation for local Clifford algebras due to Jones and Wassermann [**Wa3**] which we review in §2.2. The local equivalence and factoriality properties are proved in §2.4.

2.1. Modular theory.

We outline Tomita and Takesaki's modular theory, details may be found in [**BR**]. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with commutant M'. Notice that $\xi \in \mathcal{H}$ is *cyclic* for M, *i.e.* $\overline{M\xi} = \mathcal{H}$ iff it is *separating* for M' *i.e.* $x\xi = 0$, $x \in M'$ implies $x = 0$. Indeed, if ξ is cyclic and annihilated by $x \in M'$, then $xM\xi = Mx\xi = 0$ and therefore $x = 0$. Conversely, the orthogonal projection p on $\overline{M\xi}$ lies in M' and $(1-p)\xi = 0$ whence $p = 1$.

Assume now that $\Omega \in \mathcal{H}$ is cyclic and separating for M so that $\overline{M\Omega} = \mathcal{H} = \overline{M'\Omega}$ and $\mathcal{H}$ is the GNS representation corresponding to the (faithful and normal) state $\phi_\Omega(x) = (x\Omega, \Omega)$, $x \in M$. Let κ be the densely defined conjugation given by $\kappa x\Omega = x^*\Omega$ on $\mathcal{D}(\kappa) = M\Omega$. If ϕ_Ω is a tracial state *i.e.* $\phi_\Omega(ab) = \phi_\Omega(ba)$ then κ is an isometry and therefore extends to the whole of $\mathcal{H}$. As is easily verified, $\kappa M \kappa' \subset M'$ and von Neumann's commutation theorem asserts that in that case $\kappa M \kappa = M'$. More generally, κ is closeable with closure $\overline{\kappa} = J\Delta^{\frac{1}{2}}$ where J is a conjugate linear isometry, the phase of the polar decomposition of $\overline{\kappa}$ and the modulus $\Delta^{\frac{1}{2}}$ is a positive self-adjoint operator which measures the failure of ϕ_Ω to be tracial. Tomita's fundamental theorem asserts that $JMJ = M'$ and that, in addition $\Delta^{it} M \Delta^{-it} = M$. J and Δ^{it} are called the modular conjugation and group.

The modular group is a fundamental tool in the study of M in view of Connes' result that the class of Δ^{it} in $\operatorname{Out}(M) = \operatorname{Aut}(M)/\operatorname{Inn}(M)$ is independent of the particular pair $(\Omega, \mathcal{H})$. This implies that if Δ^{it} acts ergodically, *i.e.* without fixed points on M then M is a factor (for $Z(M)$ is fixed by $\operatorname{Ad}(\Delta^{it})$) of type III_1 for in all other cases the modular group is inner, as a suitable choice of $\mathcal{H}$ shows. An important special case arises when Δ^{it} leaves no vectors in $\mathcal{H}$ fixed aside from Ω. Then $\Delta^{it} x \Delta^{-it} = x$ for some $x \in M$ implies that $x\Omega$ is fixed by the modular group and therefore $x\Omega = \alpha\Omega$ for some $\alpha \in \mathbb{C}$. Since Ω is separating, $x \equiv \alpha$ and the modular group acts ergodically on M.

The modular operators are *hereditary* in the following sense [**Ta**]. If $N \subset M$ is a sub–von Neumann algebra such that $\Delta^{it} N \Delta^{-it} = N$, then J and Δ^{it} restrict to the modular operators for N acting on $\mathcal{K} = \overline{N\Omega}$. More precisely, if $p \in N'$ is the orthogonal projection on $\mathcal{K}$, then $y \to yp$ is an isomorphism of N onto $pNp = Np \subset \mathcal{B}(\mathcal{K})$ for $yp = 0$ implies $y\Omega = y(1-p)\Omega = 0$ and therefore $y = 0$ since Ω is separating. $\Omega \in \mathcal{K}$ is cyclic and separating for pNp and J and Δ^{it} leave $\mathcal{K}$ invariant and restrict to the modular operators for pNp relative to Ω. Following Jones and Wassermann [**Wa1, Wa2, Wa3**], we shall refer to the hereditarity of the modular operators as *Takesaki devissage*.

2.2. Modular operators for complex fermions on S^1.

We describe, following Jones and Wassermann [**Wa1, Wa3**], the modular operators for the local CAR algebras $\mathfrak{A}(L^2(I, \mathbb{C}^n))$, $I \subset S^1$ in the Fock space corresponding to the basic representation of LU_n. Their explicit knowledge will be used in §2.3 to deduce Haag duality in the vacuum representation of $L\operatorname{Spin}_{2n}$ from its fermionic realisation.

Recall that the group

$$\mathrm{SU}(1,1)_\pm = \{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \,|\, |\alpha|^2 - |\beta|^2 = \pm 1 \} \tag{2.2.1}$$

acts on $S^1 = \{z \,|\, |z| = 1\}$ by Möbius transformations. $\mathrm{SU}(1,1)_\pm$ is the semi–direct product of the connected component of the identity $\mathrm{SU}(1,1)$ by its group of components $\mathbb{Z}_2$ generated by the matrix

$$j = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{2.2.2}$$

acting on S^1 as the flip $z \to z^{-1}$. $\mathrm{SU}(1,1)$ acts by orientation preserving diffeomorphisms of the circle and maps any three points $z_1, z_2, z_3 \in S^1$ to any other three points w_1, w_2, w_3 provided z_2 and w_2 lie on the anticlockwise circular arcs going from z_1 to z_3 and w_1 to w_3 respectively. In particular, there is a one–parameter group in $\mathrm{SU}(1,1)_+$ of elements fixing the endpoints of the upper semi–circle

$\{z \in S^1 | \operatorname{Im} z \geq 0\}$ given by

$$t \to d^t = \begin{pmatrix} \cosh \pi t & \sinh \pi t \\ \sinh \pi t & \cosh \pi t \end{pmatrix} \tag{2.2.3}$$

Let now $\mathcal{H} = L^2(S^1, \mathbb{C}^n)$ with CAR algebra $\mathfrak{A}(\mathcal{H})$ generated by the $\mathbb{C}$–linear symbols $c(f)$ subject to

$$\{c(f), c(g)\} = 0 \qquad \{c(f), c(g)^*\} = (f, g) \tag{2.2.4}$$

Denote by $\mathcal{H}_+ \subset \mathcal{H}$ the space of functions with vanishing positively moded Fourier coefficents, *i.e.* the boundary values of holomorphic functions on $|z| > 1$ and by $\mathcal{H}_-$ its orthogonal complement. Let $\mathcal{J}$ be the complex structure acting as multiplication by $\pm i$ on $\mathcal{H}_\pm$ and denote by $\mathcal{H}_\mathcal{J}$ the Hilbert space $\mathcal{H}$ with complex multiplication given by $\mathcal{J}$. We consider, as in §2.2 of chapter III, the corresponding representation of $\mathfrak{A}(\mathcal{H})$ on the Fock space $\mathcal{F} = \Lambda\mathcal{H}_\mathcal{J}$. The subgroup of the unitary group $U(\mathcal{H})$ commuting with $\mathcal{J}$ is canonically quantised on $\mathcal{F}$ by

$$\Gamma(u) f_1 \wedge \cdots \wedge f_m = u f_1 \wedge \cdots \wedge u f_m \tag{2.2.5}$$

Moreover, unitaries of $\mathcal{H}$ anticommuting with $\mathcal{J}$ are canonically represented on $\mathcal{F}$ by anti–unitaries.

$SU(1,1)_\pm$ acts unitarily on $\mathcal{H}$ by

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} f(z) = \left(\frac{z}{\overline{\alpha} z - \beta} \right) f(\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}^{-1} z) \tag{2.2.6}$$

When restricted to $SU(1,1)$, this action leaves $\mathcal{H}_+$ invariant since $|\alpha| > |\beta|$ on this subgroup and therefore $z \to z(\overline{\alpha} z - \beta)^{-1}$ is holomorphic on $|z| > 1$. Thus, $SU(1,1)$ commutes with $\mathcal{J}$ and is therefore canonically quantised on $\mathcal{F}$. On the other hand, the flip (2.2.2) acts by $jf(z) = zf(z^{-1})$ and therefore anticommutes with $\mathcal{J}$ so that $SU(1,1)_- = jSU(1,1)$ acts on $\mathcal{F}$ by antiunitaries.

For any open, non dense interval $I \subset S^1$, denote by $\mathfrak{A}(I) \subset B(\mathcal{F})$ the von Neumann algebra generated by the operators $c(f)$ with f supported in I. Let now $I = (0, \pi)$ be the upper semi–circle.

Theorem 2.2.1 (Jones–Wassermann [**Wa1, Wa3**]).
(i) *The vacuum vector* $\Omega \in \Lambda^0 \mathcal{H}_\mathcal{J} \subset \mathcal{F}$ *is cyclic and separating for* $\mathfrak{A}(I)$.
(ii) *The corresponding modular group* Δ_I^{it} *is the canonical quantisation of the action of the Möbius flow* (2.2.3) *on* $\mathcal{H}$.
(iii) *The corresponding modular conjugation* J_I *is given by* $\kappa^{-1}\Gamma(j)$ *where* $\kappa = i^\epsilon$ *is the Klein transform corresponding to the natural* $\mathbb{Z}_2$*–grading* ϵ *on* $\mathcal{F}$ *and* j *is the action of the flip* (2.2.2) *on* $\mathcal{H}$.
(iv) Δ_I^{it} *leaves no vectors in* $\mathcal{F}$ *fixed aside from* Ω.

Remark. Similar results are obtained for a general $I \subsetneq S^1$ by using the action of SU$(1,1)$ on $\mathcal{F}$.

Remark. It follows at once from Tomita's theorem and the above that $\mathfrak{A}(I)' = J\mathfrak{A}(I)J = \kappa^{-1}\mathfrak{A}(I^c)\kappa$. Moreover, the $\mathfrak{A}(I)$ are type III_1 factors since Δ_I^{it} acts ergodically.

2.3. Haag duality in the vacuum sector.

Recall from chapter III that the Neveu–Schwarz Hilbert space $\mathcal{H}_{\mathrm{NS}}$ of anti–periodic, $\mathbb{C}^n$–valued functions on S^1 carries an orthogonal action of $L\,\mathrm{Spin}_{2n} \rtimes \mathrm{Rot}\, S^1$. If $\widetilde{\mathcal{J}}$ is the complex structure acting as multiplication by i on the subspace of functions with vanishing positively–moded Fourier coefficients and by $-i$ on its orthogonal complement, then by lemma III.2.4.1, $L\,\mathrm{Spin}_{2n}$ commutes with $\widetilde{\mathcal{J}}$ up to Hilbert–Schmidt operators and therefore acts on the Fock space $\mathcal{F}_{NS} = \Lambda\mathcal{H}_{\mathrm{NS},\widetilde{\mathcal{J}}}$. By proposition III.3.3.1, the corresponding positive energy representation is the sum of the level 1 vacuum and vector representations.

$\mathcal{H}_{\mathrm{NS}}$ is unitarily isomorphic to $\mathcal{H} = L^2(S^1, \mathbb{C}^n)$ considered in §2.2 via the map $f \to z^{\frac{1}{2}} f$ which identifies the complex structures $\widetilde{\mathcal{J}}$ and $\mathcal{J}$. Transporting the action of $L\,\mathrm{Spin}_{2n} \rtimes \mathrm{Rot}\, S^1$ via this identification,

we see that $L\,\mathrm{Spin}_{2n}$ acts on $\mathcal{H}$ by $\gamma f(z) = z^{\frac{1}{2}}\gamma(z)z^{-\frac{1}{2}}f(z)$ and $\mathrm{Rot}\,S^1$ by $R_\theta f(z) = e^{i\frac{\theta}{2}}f(e^{-i\theta}z)$ [1]. Denote by Γ_1 the corresponding projective representation of $L\,\mathrm{Spin}_{2n}$ on $\mathcal{F} = \Lambda\mathcal{H}_J$. More generally, $L\,\mathrm{Spin}_{2n}$ acts on $\mathcal{H}\otimes\mathbb{C}^\ell$ by $u \to u\otimes 1$ and commutes, up to Hilbert–Schmidt operators with $\mathcal{J}\otimes 1$. Since $\Lambda(\mathcal{H}\otimes\mathbb{C}^\ell_{\mathcal{J}\otimes 1}) \cong \Lambda\mathcal{H}_{\mathcal{J}}^{\otimes\ell} = \mathcal{F}^{\otimes\ell}$, the resulting projective representation Γ_ℓ is equivalent to $\Gamma_1^{\otimes\ell}$.

Let, as in 2.2, $\mathfrak{A}(I)$ be the von Neumann algebra generated in $\mathcal{B}(\mathcal{F}^{\otimes\ell})$ by the $c(f)$ with f supported in I. Notice that $\Gamma_\ell(L_I\,\mathrm{Spin}_{2n})$ commutes with $\mathfrak{A}(I^c)$ since, by (III.2.1.5) and (III.2.3.7),

$$\Gamma_\ell(\gamma)c(f)\Gamma_\ell(\gamma)^* = \frac{1}{2}\Gamma_\ell(\gamma)(\psi(f) - i\psi(if))\Gamma_\ell(\gamma)^* = \frac{1}{2}(\psi(\gamma f) - i\psi(\gamma i f)) = c(f) \tag{2.3.1}$$

whenever $\gamma \in L_I\,\mathrm{Spin}_{2n}$ and f is supported in I^c. Let Δ_I^{it} be the modular group for $\mathfrak{A}(I)$ relative to the vacuum $\Omega^{\otimes\ell}$, then

Lemma 2.3.1. *$\Gamma_\ell(L_I\,\mathrm{Spin}_{2n})'' \subset \mathfrak{A}(I)$ and is normalised by Δ_I^{it}.*

Proof. We claim that $L\,\mathrm{Spin}_{2n}$ acts by even operators, *i.e.* that Γ_ℓ commutes with the $\mathbb{Z}_2$-grading ϵ on $\mathcal{F}^{\otimes\ell}$. Indeed, ϵ is the canonical quantisation of multiplication by -1 on $\mathcal{H}\otimes\mathbb{C}^\ell$ and therefore

$$\Gamma_\ell(\gamma)\epsilon\Gamma_\ell(\gamma)^* = \chi(\gamma)\epsilon \tag{2.3.2}$$

where $\chi(\gamma) = \pm 1$ and depends continuously on γ. By connectedness of $L\,\mathrm{Spin}_{2n}$, $\chi \equiv 1$. Now, by (2.3.1), $\Gamma_\ell(L_I\,\mathrm{Spin}_{2n})$ commutes with $\mathfrak{A}(I^c)$ and therefore, by theorem 2.2.1 lies in $\mathfrak{A}(I^c)' = \kappa^{-1}\mathfrak{A}(I)\kappa$. Thus, since $[\epsilon, \Gamma_\ell(L_I\,\mathrm{Spin}_{2n})] = 0$, we get

$$\Gamma_\ell(L_I\,\mathrm{Spin}_{2n})'' = \kappa\Gamma_\ell(L_I\,\mathrm{Spin}_{2n})''\kappa^{-1} \subset \mathfrak{A}(I) \tag{2.3.3}$$

To see that $\Gamma_\ell(L_I\,\mathrm{Spin}_{2n})''$ is normalised by Δ_I^{it}, notice that if $\gamma \in L_I\,\mathrm{Spin}_{2n}$ and $A^{-1} \in SU(1,1)$ leaves I invariant, then in $PU(\mathcal{F}^{\otimes\ell})$, $\Gamma(A)\Gamma_\ell(\gamma)\Gamma(A)^* = \Gamma(A\gamma A^{-1})$. Now $A\gamma A^{-1}$ is multiplication by

$$\left(\frac{z}{\overline{\alpha}z - \beta}\right) w^{\frac{1}{2}}\gamma(A^{-1}z)w^{-\frac{1}{2}}\left(\frac{z}{\overline{\alpha}z - \beta}\right)^{-1} = z^{\frac{1}{2}}\left(\frac{z}{\overline{\alpha}z - \beta}\right)\left(\frac{w}{z}\right)^{\frac{1}{2}}\gamma(A^{-1}z)\left(\frac{w}{z}\right)^{-\frac{1}{2}}\left(\frac{z}{\overline{\alpha}z - \beta}\right)^{-1} z^{-\frac{1}{2}} \tag{2.3.4}$$

where $w = A^{-1}z$. This is the action of a loop in $L_I\,\mathrm{Spin}_{2n}$ since SO_{2n} is a normal subgroup of $M_{2n}(\mathbb{R})$ and therefore, by the explicit geometric form of the modular group given by theorem 2.2.1, $\Gamma_\ell(L_I\,\mathrm{Spin}_{2n})''$ is invariant under Δ_I^{it} $\diamond$

Remark. Notice that the inclusion $\pi(L_I\,\mathrm{Spin}_{2n})'' \subset \mathfrak{A}(I)$ and the evenness of the operators $\Gamma_\ell(\gamma)$, $\gamma \in L\,\mathrm{Spin}_{2n}$ imply that $\Gamma_\ell(L_{I^c}\,\mathrm{Spin}_{2n})'' \subset \Gamma_\ell(L_I\,\mathrm{Spin}_{2n})'$ and therefore give another proof of locality.

Lemma 2.3.2. *$\overline{\Gamma_\ell(L_I\,\mathrm{Spin}_{2n})''\Omega^{\otimes\ell}} \subset \mathcal{F}^{\otimes\ell}$ is the vacuum representation of $L\,\mathrm{Spin}_{2n}$ at level ℓ.*

Proof. Let $\mathcal{K} = \overline{\Gamma_\ell(L\,\mathrm{Spin}_{2n})''\Omega^{\otimes\ell}}$. $\mathcal{K}$ is an irreducible $L\,\mathrm{Spin}_{2n} \rtimes \mathrm{Rot}\,S^1$ module since if $\mathcal{W} \subset K$ is a submodule and P is the corresponding orthogonal projection then $\mathcal{W} = \overline{\Gamma_\ell(L\,\mathrm{Spin}_{2n})''P\Omega^{\otimes\ell}}$. Since P commutes with $\mathrm{Rot}\,S^1$, $P\Omega^{\otimes\ell} = \lambda\Omega^{\otimes\ell}$ with $\lambda \in \{0,1\}$ since the latter is the only vector in $\mathcal{F}^{\otimes\ell}$ fixed by $\mathrm{Rot}\,S^1$ [2]. It follows that $\mathcal{W} = 0$ or $\mathcal{K}$. Since $\mathcal{K}$ is of positive energy, it is uniquely characterised by its lowest energy subspace and therefore isomorphic to the vacuum representation of level ℓ. The claim now follows from the Reeh–Schlieder theorem $\diamond$

Proposition 2.3.3 (Haag duality in the vacuum sector). *Let $(\pi, \mathcal{H}_0)$ be the vacuum representation at level ℓ. Then,*

$$\pi_0(L_I\,\mathrm{Spin}_{2n})' = \pi_0(L_{I^c}\,\mathrm{Spin}_{2n})'' \tag{2.3.5}$$

[1] This is not unitarily equivalent to the action on $\mathcal{H}$ given by $\gamma f(z) = \gamma(z)f(z)$ since Spin_{2n} does not commute with the complex structure on $\mathbb{C}^n$. The latter action is easily recognised to be that of the Ramond sector whose quantisaton leads to the two level 1 spin representations of $L\,\mathrm{Spin}_{2n}$

[2] We are referring to the action of $\mathrm{Rot}\,S^1$ transported from $\mathcal{H}_{\mathrm{NS}}$ **not** to the natural one on $\mathcal{H}$

Proof. By lemma 2.3.1 and Takesaki devissage, the modular conjugation J_I of $\mathfrak{A}(I)$ restricts to the one for $\Gamma_\ell(L_I \operatorname{Spin}_{2n})''$ on $\overline{\Gamma_\ell(L_I \operatorname{Spin}_{2n})''\Omega^{\otimes\ell}} \subset \mathcal{F}^{\otimes\ell}$ which, by lemma 2.3.2 is isomorphic to $\mathcal{H}_0$. When I is the upper semi–circle, Haag duality follows from (iii) of theorem 2.2.1 because, in $PU(\mathcal{F}^{\otimes\ell})$,

$$J\Gamma_\ell(\gamma)J = \kappa^{-1}\Gamma(j)\Gamma_\ell(\gamma)\Gamma(j)^*\kappa = \kappa^{-1}\Gamma(j\gamma j)\kappa \tag{2.3.6}$$

and $j\gamma j$ is multiplication by $z^{\frac{1}{2}}\gamma(z^{-1})z^{-\frac{1}{2}}$ which lies in $L_{I^c} \operatorname{Spin}_{2n}$. For a general I, the result follows by conjugating by an appropriate element of $SU(1,1)$ $\diamond$

Remark. Since conjugation by discontinuous loops normalises each $L_I \operatorname{Spin}_{2n}$, Haag duality holds for any representation obtained by conjugating π_0 by an element of $L_{Z(\operatorname{Spin}_{2n})} \operatorname{Spin}_{2n}$ and in particular for all level 1 representations of $L \operatorname{Spin}_{2n}$.

2.4. Local equivalence and factoriality.

Proposition 2.4.1 (Local equivalence). *All irreducible positive energy representations of level ℓ are unitarily equivalent for the local loop groups $L_I \operatorname{Spin}_{2n}$.*

Proof. For $\ell = 1$, local equivalence follows by conjugating by localised discontinuous loops. Indeed, by corollary I.3.2.5, the action of $L_{Z(\operatorname{Spin}_{2n})} \operatorname{Spin}_{2n}$ on the irreducible representations of level 1 is transitive since Spin_{2n} is simply–laced. Thus, if $\mathcal{H}_1$ is obtained from $\mathcal{H}_2$ by conjugating by discontinuous loops in a given $L \operatorname{Spin}_{2n}$–coset, choosing a representative equal to 1 on I yields a unitary equivalence of $\mathcal{H}_1$ and $\mathcal{H}_2$ as $L_I \operatorname{Spin}_{2n}$–modules.
For a general ℓ, we shall need the fact that $M = \Gamma_\ell(L_I \operatorname{Spin}_{2n})''$ is a factor of type III_1. Indeed, by lemma 2.3.1 and Takesaki devissage, Δ_I^{it} is the modular group of M relative to $\Omega^{\otimes\ell}$ which, by (iv) of theorem 2.2.1 acts ergodically on M thus proving our claim. Now, if π_0 is the level 1 vacuum representation of $L \operatorname{Spin}_{2n}$, then $N = \pi_0^{\otimes\ell}(L_I \operatorname{Spin}_{2n})''$ is also a type III_1 factor. Indeed, $\pi_0^{\otimes\ell}$ is a subrepresentation of Γ_ℓ so that if $p \in M'$ is the corresponding orthogonal projection, $N = pMp = Mp$. Moreover, the map $x \to xp$ is an isomorphism since, by the Reeh–Schlieder theorem, any finite energy vector $\xi \in \pi_0^{\otimes\ell}$ is separating for M and therefore $xp = 0$ implies $x\xi = x(1-p)\xi = 0$. Thus, $\pi_0^{\otimes\ell}(L_I \operatorname{Spin}_{2n})''$ is a factor of type III and therefore so is N'. It follows that all subrepresentations of $\pi_0^{\otimes\ell}$ are isomorphic since they correspond to the projections in N' and these are all equivalent.
To conclude, note that by local equivalence at level 1, $\pi_0^{\otimes\ell}$ is locally equivalent to $\pi_{i_1} \otimes \cdots \otimes \pi_{i_\ell}$ where the π_{i_j} are any level 1 representations. By proposition I.2.3.3, any irreducible positive energy representation of $L \operatorname{Spin}_{2n}$ at level ℓ is a summand of some such tensor product and our claim is established $\diamond$

The proof of proposition 2.4.1 has the following important

Corollary 2.4.2 (Factoriality). *The von Neumann algebras generated in a positive energy representation by the local loop groups $L_I \operatorname{Spin}_{2n}$ are factors of type III_1.*

Remark.

(i) Proposition 2.4.1 implies that the local von Neumann algebras $\pi_i(L_I \operatorname{Spin}_{2n})''$ generated in two positive energy representations $(\pi_i, \mathcal{H}_i)$ of equal level are spatially and canonically isomorphic. Indeed, it yields the existence of a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying $U\pi_1(\gamma)U^* = \pi_2(\gamma)$ in $PU(\mathcal{H}_2)$ for any $\gamma \in L_I \operatorname{Spin}_{2n}$. If U' is another such unitary and $V = U'^*U$, then, in $U(\mathcal{H}_1)$,

$$V\pi_1(\gamma)V^* = \chi(\gamma)\pi_1(\gamma) \tag{2.4.1}$$

where $\chi(\gamma) \in \mathbb{T}$ is a character of $L_I \operatorname{Spin}_{2n}$ and is therefore trivial by lemma 1.1.1. Thus, in $U(\mathcal{H}_2)$ $U\pi_1(\gamma)U^* = U'V\pi_1(\gamma)V^*U'^* = U'\pi_1(\gamma)U'^*$ and

$$x \longrightarrow UxU^* \tag{2.4.2}$$

is a canonical spatial isomorphism $\pi_1(L_I \operatorname{Spin}_{2n})'' \cong \pi_2(L_I \operatorname{Spin}_{2n})''$.

(ii) Notice that (2.4.2) yields an isomorphism of the central extensions of $L_I \operatorname{Spin}_{2n}$ induced by π_1 and π_2. Since $\pi_i(L_I \operatorname{Spin}_{2n})$ and $\pi_i(L_{I^c} \operatorname{Spin}_{2n})$ commute by locality, this isomorphism extends to one of the central extensions of $L_I \operatorname{Spin}_{2n} \times L_{I^c} \operatorname{Spin}_{2n}$ corresponding to π_1 and

π_2. Using proposition 1.3.2, it is easy to show that it extends futher to one of central extensions of $L\,\mathrm{Spin}_{2n}$ thus giving an alternative proof of (ii) of proposition II.2.4.3.

CHAPTER V

Bosonic construction of level 1 representations and primary fields

This chapter is devoted to the bosonic or Frenkel–Kac–Segal vertex operator construction of the level 1 representations of $L\,\mathrm{Spin}_{2n}$. This stems from the *a posteriori* observation that they remain irreducible when restricted to the abelian group LT of loops in a maximal torus $T \subset \mathrm{Spin}_{2n}$. A suitable Stone–von Neumann theorem may then be used to reconstruct them from the Heisenberg representations of LT. Our interest in this construction lies in the fact that it gives a particularly convenient description of all level 1 primary fields which will be used in chapter VI to establish their continuity properties as operator–valued distributions.

The vertex operator construction applies in fact to the loop groups of all compact, simply–connected, simple and simply–laced Lie groups G and we shall treat it in this generality. In section 2, we study the restriction of positive energy representations of LG to the loop group $LT = C^\infty(S^1, T)$ of a maximal torus $T \subset G$ and compute the corresponding projective cocycle on LT. As is well–known, the associated central extension factors as a product of two Heisenberg groups, one corresponding to the identity component of LT/T, the other to the product of the constant loops $T \subset LT$ by the integral lattice $\check{T} = \mathrm{Hom}(\mathbb{T}, T)$. Suitable versions of the Stone–von Neumann theorem classifying the irreducible representations of these groups are obtained in section 3.

We also compute in section 2 the commutation relations of LT with the infinitesimal action of $L^{\mathrm{pol}}\mathfrak{g}$ in a positive energy representation. In section 3, we construct operators in the irreducible representations of LT which mimick $L^{\mathrm{pol}}\mathfrak{g}$ in that they have the same commutation relations with LT. We then show in section 4 that they give level 1 representations of $L^{\mathrm{pol}}\mathfrak{g}$. The reader familiar with the vertex operator construction may skip to section 5 where the construction of the level 1 primary fields is carried out by following a similar scheme. The equivariance properties of the primary fields with respect to LT are obtained and operators satisfying these constructed explicitly which, by inspection have the required commutation relations with $L^{\mathrm{pol}}\mathfrak{g}$.

1. Formal variables

We begin by briefly reviewing the formal variable approach of Frenkel, Lepowski and Meurman [**FLM**, chap. 2]. This gives a convenient way to encode the infinitesimal action of $L^{\mathrm{pol}}\mathfrak{g}$ on the finite energy subspace $\mathcal{H}^{\mathrm{fin}}$ of a positive energy representation of LG.

Let ℓ be the level of $\mathcal{H}$. The commutation relations satisfied by $L^{\mathrm{pol}}\mathfrak{g}$ on $\mathcal{H}^{\mathrm{fin}}$, namely

$$[X(m), Y(n)] = [X, Y](m+n) + \ell m \delta_{m+n,0}\langle X, Y\rangle \tag{1.1}$$

may equivalently be written in terms of the generating function $X(z) \in \mathrm{End}(\mathcal{H}^{\mathrm{fin}})[[z, z^{-1}]]$ defined by

$$X(z) = \sum_{n\in\mathbb{Z}} X(n) z^{-n} \tag{1.2}$$

as

$$\begin{aligned}[X(z),Y(\zeta)] &= \sum_{m,n}[X,Y](m+n)\zeta^{-(m+n)}\Big(\frac{\zeta}{z}\Big)^m + \ell\langle X,Y\rangle\sum_m m\Big(\frac{\zeta}{z}\Big)^m \\ &= [X,Y](\zeta)\delta\Big(\frac{\zeta}{z}\Big) + \ell\langle X,Y\rangle\delta'\Big(\frac{\zeta}{z}\Big)\end{aligned} \tag{1.3}$$

where

$$\delta(u)=\sum_{n\in\mathbb{Z}}u^n=\delta(u^{-1}) \qquad \text{and} \qquad \delta'(u)=\sum_{n\in\mathbb{Z}}nu^n=-\delta'(u^{-1}) \tag{1.4}$$

are the formal Dirac delta function and its first derivative. Similarly, the relation $[d,X(n)]=-nX(n)$ and the formal adjunction property $X(n)^*=-\overline{X}(-n)$ are equivalent to

$$[d,X(z)] = z\frac{d}{dz} \tag{1.5}$$

$$X(z)^* = -\overline{X}(z) \tag{1.6}$$

where the adjoint of $A(z)=\sum_{n\in\mathbb{Z}}A(n)z^{-n}$ is defined whenever the modes $A(n)$ possess formal adjoints by $\sum_{n\in\mathbb{Z}}A(n)^*z^n$ so that it satisfies (1.5) if, and only if $A(z)$ does. We will need, for later reference, the following elementary

LEMMA 1.1. *Let u be a formal variable and define, for any $a\in\mathbb{C}$*

$$(1-u)^a=\sum_{n\geq 0}\binom{a}{n}(-u)^n \tag{1.7}$$

Then

$$\delta(u)=(1-u)^{-1}+(1-u^{-1})^{-1}u^{-1} \tag{1.8}$$

$$\delta'(u)=u(1-u)^{-2}-(1-u^{-1})^{-2}u^{-1} \tag{1.9}$$

LEMMA 1.2. *Set $u=\frac{\zeta}{z}$ for two formal variables ζ and z and regard $\delta(u)$ and $\delta'(u)$ as formal Laurent series in ζ,z. Then, for any $f(z,\zeta)\in\mathbb{C}[[z,\zeta,z^{-1},\zeta^{-1}]]$ such that $f(\zeta,\zeta)$ is a well–defined formal Laurent series,*

$$f(z,\zeta)\delta\Big(\frac{\zeta}{z}\Big)=f(\zeta,\zeta)\delta\Big(\frac{\zeta}{z}\Big)=f(z,z)\delta\Big(\frac{\zeta}{z}\Big) \tag{1.10}$$

$$f(z,\zeta)\delta'\Big(\frac{\zeta}{z}\Big)=f(\zeta,\zeta)\delta'\Big(\frac{\zeta}{z}\Big)+z\frac{\partial f}{\partial z}(\zeta,\zeta)\delta\Big(\frac{\zeta}{z}\Big)=f(z,z)\delta'\Big(\frac{\zeta}{z}\Big)-\zeta\frac{\partial f}{\partial\zeta}(z,z)\delta\Big(\frac{\zeta}{z}\Big) \tag{1.11}$$

REMARK. It is important to assume that $f(z,\zeta)$ only contains integral powers of z,ζ in the above lemma since $f(z,\zeta)=\big(\frac{\zeta}{z}\big)^{\frac{1}{2}}$ clearly does not satisfy (1.10).

2. Restriction of positive energy representations to LT

Unlike the quark model of chapter III, the vertex operator construction is purely infinitesimal and gives an explicit description of the action of $L^{\text{pol}}\mathfrak{g}$ on the finite energy subspaces of level 1 representations only. These subspaces are studied as LT–modules in this section and will be reconstructed as such in section 4. Strictly speaking, they are not invariant under the identity component of LT/T and we shall trade the latter for its algebraic Lie algebra $L^{\text{pol}}\mathfrak{t}/\mathfrak{t}$ consisting of polynomial loops in $\mathfrak{t}=\text{Lie}(T)$ with zero average.

2.1. The loop group LT.

Let $T\subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$ and integral lattice $I=\{h\in\mathfrak{t}|\ \exp_T(2\pi h)=1\}$. We denote by $LT=C^\infty(S^1,T)$ the loop group of T. Since

$$0\to 2\pi I\to\mathfrak{t}\xrightarrow{\exp}T\to 1 \tag{2.1.1}$$

is the universal covering group of T, we have

$$LT \cong \{f \in C^\infty([0,2\pi],\mathfrak{t}) | f(2\pi) - f(0) \in 2\pi I\}/2\pi I = \bigsqcup_{\lambda\in I} LT_\lambda \tag{2.1.2}$$

where LT_λ is the connected component of loops f whose winding number $\nu_f = \frac{1}{2\pi}\Big(f(2\pi) - f(0)\Big)$ equals $\lambda \in I$. We therefore have an isomorphism $LT \cong L\mathfrak{t}/\mathfrak{t} \times T \times \check{T}$, where $L\mathfrak{t}/\mathfrak{t}$ is the additive group of smooth loops in $\mathfrak{t}$ of zero average and $\check{T} = \mathrm{Hom}(\mathbb{T}, T)$, given by

$$f \rightarrow \Big(f - f_0 - \nu_f(\theta - \pi), \exp_T(f_0), \exp_T(\theta\nu_f)\Big) \tag{2.1.3}$$

where $f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$. Since G is simply–connected, multiplication by i yields an identification of the integral lattice I with the coroot lattice $\Lambda_R^\vee \subset i\mathfrak{t}$ [**Ad**, thm. 5.47]. The simple–lacedness of G implies in turn that the latter is isomorphic to the root lattice $\Lambda_R \subset i\mathfrak{t}^*$ via the identification $i\mathfrak{t} \cong i\mathfrak{t}^*$ defined by the basic inner product $\langle\cdot,\cdot\rangle$. Using these identifications, we shall parametrise elements of $\check{T}$ by associating to each $\alpha \in \Lambda_R$ the homomorphism

$$\zeta_\alpha(\theta) = \exp_T(-i\alpha\theta) \tag{2.1.4}$$

2.2. Commutation relations of LT and $L^{\mathrm{pol}}\mathfrak{g}$.

Let $(\pi, \mathcal{H})$ be a level ℓ positive energy representation of LG and denote by $U_\theta = e^{id\theta}$ the corresponding integrally–moded action of $\mathrm{Rot}\, S^1$. π lifts uniquely over $G \subset LG$ to a unitary representation commuting with U_θ which we denote by the same symbol. Recall from chapter II that the action of LG leaves the subspace of smooth vectors $\mathcal{H}^\infty$ invariant and, by theorem II.1.6.3 satisfies

$$\pi(\gamma)\pi(X)\pi(\gamma)^* = \pi(\gamma X\gamma^{-1}) - i\ell\int_0^{2\pi}\langle\gamma^{-1}\dot{\gamma}, X\rangle\frac{d\theta}{2\pi} \tag{2.2.1}$$

$$\pi(\gamma)d\pi(\gamma)^* = d - i\pi(\dot{\gamma}\gamma^{-1}) - \frac{\ell}{2}\int_0^{2\pi}\langle\gamma^{-1}\dot{\gamma}, \gamma^{-1}\dot{\gamma}\rangle\frac{d\theta}{2\pi} \tag{2.2.2}$$

for any $\gamma \in LG$ and $X \in L\mathfrak{g}$.

Proposition 2.2.1. *The action of $T \times \check{T}$ leaves the subspace of finite energy vectors $\mathcal{H}^{\mathrm{fin}}$ invariant and satisfies, for any $\tau \in T$ and $\alpha \in \Lambda_R$*

$$[d, \pi(\tau)] = 0 \tag{2.2.3}$$

$$[d, \pi(\zeta_\alpha)] = \pi(\zeta_\alpha)(\pi(\alpha) + \frac{\langle\alpha,\alpha\rangle}{2}) \tag{2.2.4}$$

Proof. The action of T leaves $\mathcal{H}^{\mathrm{fin}}$ invariant since $\pi(g)U_\theta\pi(g)^* = U_\theta$ for any $g \in G$. If $\alpha \in \Lambda_R$ then, projectively

$$\pi(\zeta_\alpha)U_\theta\pi(\zeta_\alpha)^* = \pi(\zeta_\alpha(\zeta_\alpha)_\theta^{-1})U_\theta = \pi(\exp_T(-i\alpha\theta))U_\theta \tag{2.2.5}$$

Thus, $\theta \rightarrow \pi(\exp_T(-i\alpha\theta))U_\theta$ is a positive energy action of $\mathrm{Rot}\, S^1$. Since it commutes with U_θ, their finite energy subspaces coincide and therefore $\pi(\zeta_\alpha)\mathcal{H}^{\mathrm{fin}} = \mathcal{H}^{\mathrm{fin}}$. The relations (2.2.3)–(2.2.4) follow at once from (2.2.2) $\diamond$

Proposition 2.2.2. *The action of $T \times \check{T}$ on $\mathcal{H}^{\mathrm{fin}}$ satisfies, for any $X \in L^{\mathrm{pol}}\mathfrak{g}$*

$$\pi(\tau)\pi(X)\pi(\tau)^* = \pi(\tau X\tau^{-1}) \tag{2.2.6}$$

$$\pi(\zeta_\alpha)\pi(X)\pi(\zeta_\alpha)^* = \pi(\zeta_\alpha X\zeta_\alpha^{-1}) - \ell\int_0^{2\pi}\langle\alpha, X\rangle\frac{d\theta}{2\pi} \tag{2.2.7}$$

In particular,

$$\pi(\tau)\pi(\zeta_\alpha)\pi(\tau)^* = \alpha(\tau)^\ell\pi(\zeta_\alpha) \tag{2.2.8}$$

where $\alpha(\exp_T(h)) = e^{\langle\alpha,h\rangle}$. Moreover, if $X_\beta \in \mathfrak{g}_\beta$ and $X_\beta(z)$ is given by (1.2), then

$$\pi(\tau)X_\beta(z)\pi(\tau)^* = \beta(\tau)X_\beta(z) \tag{2.2.9}$$

$$\pi(\zeta_\alpha)X_\beta(z)\pi(\zeta_\alpha)^* = z^{-\langle\alpha,\beta\rangle}X_\beta(z) \tag{2.2.10}$$

Proof. The relations (2.2.6), (2.2.7) and (2.2.9) follow at once from (2.2.1). Since T and $\check{T}$ commute projectively, the following holds in $U(\mathcal{H})$ for any $h \in \mathfrak{t}$,

$$\pi(\zeta_\alpha)e^{t\pi(h)}\pi(\zeta_\alpha)^* = e^{tc}e^{t\pi(h)} \tag{2.2.11}$$

for some $c \in i\mathbb{R}$. Applying both sides to $\xi \in \mathcal{H}^{\text{fin}}$ and taking derivatives at $t=0$ yields, by comparison with (2.2.7), $c = -\ell\langle\alpha, h\rangle$ and therefore (2.2.8). Let now $X_\beta \in \mathfrak{g}_\beta$ so that $[h, X_\beta] = \langle h, \beta\rangle X_\beta$ for any $h \in \mathfrak{t}$. Then, $\text{Ad}(\exp_T(h))X_\beta = e^{\langle h,\beta\rangle}X_\beta$ and therefore, in $L\mathfrak{g}$,

$$\zeta_\alpha X_\beta(n)\zeta_\alpha^{-1}(\theta) = e^{n-i\langle\alpha,\beta\rangle\theta}X_\beta = X_\beta(n - \langle\alpha,\beta\rangle) \tag{2.2.12}$$

Since $\zeta_\alpha^{-1}\dot{\zeta}_\alpha = -i\alpha \in \mathfrak{t}$ and this subspace is orthogonal to $\mathfrak{g}_\beta$ with respect to the Killing form, it follows from (2.2.1) that

$$\pi(\zeta_\alpha)\pi(X_\beta(n))\pi(\zeta_\alpha)^* = \pi(X_\beta(n - \langle\alpha,\beta\rangle)) \tag{2.2.13}$$

and therefore that (2.2.10) holds $\diamond$

2.3. The projective cocycle on $\check{T}$.

The following result is due to Pressley and Segal [**PS**, prop. 4.8.1]

Proposition 2.3.1. *Let $(\pi, \mathcal{H})$ be a positive energy representation of LG at level ℓ. Then, for any $\alpha, \beta \in \Lambda_R$,*

$$\pi(\zeta_\alpha)\pi(\zeta_\beta)\pi(\zeta_\alpha)^*\pi(\zeta_\beta)^* = (-1)^{\ell\langle\alpha,\beta\rangle} \tag{2.3.1}$$

We shall derive proposition 2.3.1 from a somewhat more general result which will be needed in section 5. Recall that the isomorphism $i\mathfrak{t}^* \cong i\mathfrak{t}$ corresponding to the basic inner product $\langle\cdot,\cdot\rangle$ identifies the weight and coweight lattices $\Lambda_W, \Lambda_W^\vee$ since G is simply–laced. In particular, any $\mu \in \Lambda_W$ defines, in the terminology of section I.3, a discontinuous loop $\zeta_\mu(\theta) = \exp_T(-i\mu\theta)$ which acts on $L\mathfrak{g}$ by conjugation.

Proposition 2.3.2. *Let $(\pi, \mathcal{H})$ be a level ℓ positive energy representation of LG. Assume that the finite energy subspace $\mathcal{H}^{\text{fin}} \subset \mathcal{H}$ supports a projective unitary representation ρ of an intermediate lattice $\Lambda_R \subset \Lambda \subset \Lambda_W$ satisfying, for any $X \in L^{\text{pol}}\mathfrak{g}$*

$$\rho(\mu)\pi(X)\rho(\mu)^* = \pi(\zeta_\mu X\zeta_\mu^{-1}) - \ell\int_0^{2\pi}\langle\mu, X\rangle\frac{d\theta}{2\pi} \tag{2.3.2}$$

Then, for any $\alpha \in \Lambda_R$ and $\mu \in \Lambda$

$$\rho(\mu)\pi(\zeta_\alpha)\rho(\mu)^*\pi(\zeta_\alpha)^* = (-1)^{\ell\langle\mu,\alpha\rangle} \tag{2.3.3}$$

Proof of proposition 2.3.1. Let $\Lambda = \Lambda_R$ in proposition 2.3.2 and ρ the restriction of π to $\Lambda_R \cong \check{T}$. By proposition 2.2.1, ρ leaves $\mathcal{H}^{\text{fin}}$ invariant and, by (2.2.7) satisfies (2.3.2). The conclusion therefore follows from proposition 2.3.2 $\diamond$

Proof of proposition 2.3.2. Notice first that (2.3.3) holds if $\mu = -\alpha$. Indeed, by proposition 2.2.1 and (2.2.7), the unitary $T_\alpha = \pi(\zeta_\alpha)\rho(-\alpha)$ leaves $\mathcal{H}^{\text{fin}}$ invariant and commutes with the action of $L^{\text{pol}}\mathfrak{g}$. By proposition I.1.2.3 it projectively commutes with LG and therefore, for any $\gamma \in LG$, the following holds in $U(\mathcal{H})$

$$T_\alpha\pi(\gamma)T_\alpha^* = \chi(\gamma)\pi(\gamma) \tag{2.3.4}$$

for some $\chi(\gamma) \in \mathbb{T}$ which defines a character of LG. By lemma IV.1.1.1, $\text{Hom}(LG, \mathbb{T}) = 1$ and therefore $\chi \equiv 1$. In particular,

$$\rho(-\alpha)\pi(\zeta_\alpha)\rho(-\alpha)^*\pi(\zeta_\alpha)^* = \rho(-\alpha)\pi(\zeta_\alpha)T_\alpha^* = \rho(-\alpha)T_\alpha^*\pi(\zeta_\alpha) = 1 \tag{2.3.5}$$

which coincides with the right–hand side of (2.3.3) since Λ_R is an even lattice when G simply–laced.

We now establish (2.3.3) when α is a positive root and $\mu \in \Lambda$ is such that $\langle\mu, \alpha\rangle \in \{0, 1\}$. The homomorphism $\zeta_\alpha(\theta) = \exp(-i\theta\alpha)$ may be written as a product of two exponentials in LG [**PS**, 4.8.1], namely

$$\zeta_\alpha = \exp_{LG}\Big(-\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\Big)\exp_{LG}\Big(\frac{\pi}{2}(e_\alpha(1) - f_\alpha(-1))\Big) \tag{2.3.6}$$

To see this, consider the loop $\sigma_\alpha : SU_2 \to G$ corresponding to the $\mathfrak{sl}_2(\mathbb{C})$–subalgebra of $\mathfrak{g}_{\mathbb{C}}$ spanned by $e_\alpha, f_\alpha, h_\alpha = \alpha$. This induces a homomorphism $LSU_2 \to LG$ in the obvious way mapping

$$\theta \to \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{2.3.7}$$

to ζ_α. Using σ_α and the standard basis of $\mathfrak{sl}_2(\mathbb{C})$ given by

$$e_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3.8}$$

(2.3.6) reduces to a matrix check.

If $h \in \mathfrak{t}$, then $[h, e_\alpha] = \langle h, \alpha \rangle e_\alpha$ whence $\mathrm{Ad}(\exp_T(h))e_\alpha = e^{\langle h,\alpha \rangle} e_\alpha$. Therefore, since $\zeta_\mu(\theta) = \exp_T(-i\theta\mu)$, we have $\zeta_\mu e_\alpha(n)\zeta_\mu^{-1}(\theta) = e_\alpha \otimes e^{i\theta(n-\langle \alpha,\mu \rangle)} = e_\alpha(n - \langle \alpha, \mu \rangle)(\theta)$. In other words,

$$\zeta_\mu e_\alpha(n)\zeta_\mu^{-1} = e_\alpha(n - \langle \alpha, \mu \rangle) \tag{2.3.9}$$

$$\zeta_\mu f_\alpha(n)\zeta_\mu^{-1} = f_\alpha(n + \langle \alpha, \mu \rangle) \tag{2.3.10}$$

Since $-i\mu \in \mathfrak{t}$ and this subspace is orthogonal to $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with respect to the Killing form, (2.3.2) yields

$$\rho(\mu)\pi(e_\alpha(n))\rho(\mu)^* = \pi(e_\alpha(n - \langle \alpha, \mu \rangle)) \tag{2.3.11}$$

$$\rho(\mu)\pi(f_\alpha(n))\rho(\mu)^* = \pi(f_\alpha(n + \langle \alpha, \mu \rangle)) \tag{2.3.12}$$

Thus, (2.3.3) holds if $\langle \alpha, \mu \rangle = 0$. If, on the other hand $\langle \alpha, \mu \rangle = 1$, then by (2.3.6)

$$\begin{aligned}
\rho(\mu)\pi(\zeta_\alpha)\rho(\mu)^*\pi(\zeta_\alpha)^* =& \exp\Big(-\frac{\pi}{2}\pi(e_\alpha(-1) - f_\alpha(1))\Big)\exp\Big(\frac{\pi}{2}\pi(e_\alpha(0) - f_\alpha(0))\Big) \\
&\cdot \exp\Big(-\frac{\pi}{2}\pi(e_\alpha(1) - f_\alpha(-1))\Big)\exp\Big(\frac{\pi}{2}\pi(e_\alpha(0) - f_\alpha(0))\Big) \\
=& \exp\Big(-\frac{\pi}{2}\pi(e_\alpha(-1) - f_\alpha(1))\Big) \\
&\cdot \exp\Big(\frac{\pi}{2}\pi(e_\alpha(0) - f_\alpha(0))\Big)\exp\Big(-\frac{\pi}{2}\pi(e_\alpha(1) - f_\alpha(-1))\Big)\exp\Big(\frac{\pi}{2}\pi(e_\alpha(0) - f_\alpha(0))\Big)^* \\
&\cdot \exp\Big(\pi\pi(e_\alpha(0) - f_\alpha(0))\Big)
\end{aligned} \tag{2.3.13}$$

where all exponentials are defined using the spectral theorem. As is easily checked using σ_α, we have

$$\exp_{LG}\Big(\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\Big)(e_\alpha(1) - f_\alpha(-1))\exp_{LG}\Big(\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\Big)^{-1} = e_\alpha(-1) - f_\alpha(1) \tag{2.3.14}$$

Moreover, since we are conjugating by a constant loop, no correction term arises from (2.2.1) and it follows that (2.3.13) is equal to

$$\exp\Big(-\pi\pi(e_\alpha(-1) - f_\alpha(1))\Big)\exp\Big(\pi\pi(e_\alpha(0) - f_\alpha(0))\Big) \tag{2.3.15}$$

To proceed, we seek to diagonalise the above unitaries. This is best done in LSU_2 using the identity

$$e_\alpha(-m) - f_\alpha(m) = V(m)ih_\alpha(0)V(m)^* \tag{2.3.16}$$

where $V(m) \in LSU_2$ is given by $\theta \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & ie^{-im\theta} \\ ie^{im\theta} & 1 \end{pmatrix}$. Since

$$V^{-1}(m)\dot{V}(m) = \frac{m}{2}\Big(ih_\alpha(0) + e_\alpha(-m) - f_\alpha(m)\Big) \tag{2.3.17}$$

we have

$$\int_0^{2\pi} \langle V^{-1}(m)\dot{V}(m), ih_\alpha(0)\rangle \frac{d\theta}{2\pi} = -\frac{m}{2}\|h_\alpha\|^2 = -m \tag{2.3.18}$$

since G is simply–laced. Therefore, using (2.2.1), (2.3.15) is equal to

$$\begin{aligned} &e^{i\pi\ell}\pi(V(1))\exp\Big(-\pi\pi(ih_\alpha(0))\Big)\pi(V(1))^*\pi(V(0))\exp\Big(\pi\pi(ih_\alpha(0))\Big)\pi(V(0))^* \\ =&e^{i\pi\ell}\pi(V(1))\pi(\exp_{LG}(-\pi ih_\alpha(0)))\pi(V(1))^*\pi(V(0))\pi(\exp_{LG}(\pi ih_\alpha(0)))\pi(V(0))^* \end{aligned} \tag{2.3.19}$$

Since $\exp_{SU_2}(-i\pi h_\alpha(0))=-1$ lies in the centre of any central extension of $L\,\mathrm{SU}_2$, the above is equal to $(-1)^\ell=(-1)^{\ell\langle\alpha,\mu\rangle}$.

To summarise, (2.3.3) holds if α is a simple root and $\mu\in\Lambda$ equals $-\alpha$ or satisfies $\langle\mu,\alpha\rangle=0,1$ and therefore if $\mu\in\Lambda$ is the opposite of a simple root or a minimal dominant weight. Since each Λ_R coset in Λ contains a minimal dominant weight, such α, μ span Λ_R and Λ respectively and it follows that ρ and the restriction of π to $\check{T}$ projectively commute. Thus,

$$\rho(\mu)\pi(\zeta_\alpha)\rho(\mu)^*\pi(\zeta_\alpha)^*=\omega(\mu,\alpha) \tag{2.3.20}$$

where $\omega(\mu,\alpha)\in\mathbb{T}$ is easily seen to be $\mathbb{Z}$–bilinear in each of its arguments. Since $\omega(\mu,\alpha)=(-1)^{\ell\langle\mu,\alpha\rangle}$ for a spanning set of μ and α, (2.3.3) follows $\diamond$

3. Level 1 representations of LT

We classify below the irreducible projective unitary representations of LT, or more precisely $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}\times(T\times\check{T})$ satisfying the level 1 relations computed in the previous section and show that, like the level 1 representations of LG they are parametrised by the dual of $Z(G)$. Notice that, by proposition 2.2.2,

$$\pi(\tau)\pi(h(n))\pi(\tau)^*=\pi(h(n)) \tag{3.1}$$

$$\pi(\zeta_\alpha)\pi(h(n))\pi(\zeta_\alpha)^*=\pi(h(n))-\delta_{n,0}\langle\alpha,h\rangle \tag{3.2}$$

so that $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}$ and $T\times\check{T}$ commute and we may therefore study the factors separately. In each case, we introduce the vertex operators of Segal and Kac–Frenkel. These are formal Laurent series having the same commutation relations with $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}\times T\times\check{T}$ as the generating functions for a Cartan–Weyl basis of $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$. In section 4, we shall show that their modes give, together with the action of $L^{\mathrm{pol}}\mathfrak{t}$ all level 1 representations of $L^{\mathrm{pol}}\mathfrak{g}$.

3.1. Stone–von Neumann theorem and vertex operators $T\times\check{T}$.

When restricted to $T\times\check{T}\subset LT$, a positive energy representation has a somewhat hybrid nature. On the one hand, it possesses a lift to a unitary representation of T which is uniquely determined by the requirement that it should extend to one of G. On the other, proposition 2.3.1 implies that it is genuinely projective on $\check{T}$. We shall accordingly consider projective unitary representations π of $T\times\check{T}$ with a preferred lift over T which we denote by the same symbol. These will be required to satisfy (2.2.8) and (2.3.1) for $\ell=1$, namely

$$\pi(\tau)\pi(\zeta_\alpha)\pi(\tau)^*=\alpha(\tau)\pi(\zeta_\alpha) \tag{3.1.1}$$

$$\pi(\zeta_\alpha)\pi(\zeta_\beta)\pi(\zeta_\alpha)^*\pi(\zeta_\beta)^*=(-1)^{\langle\alpha,\beta\rangle} \tag{3.1.2}$$

Two such $(\pi_i,\mathcal{H}_i)$, $i=1,2$ will be regarded as unitarily equivalent if there exists a unitary $U:\mathcal{H}_1\rightarrow\mathcal{H}_2$ such that

$$U\pi_1(\tau)U^*=\pi_2(\tau)\text{ in }U(\mathcal{H}_2)\qquad\text{and}\qquad U\pi_1(\zeta_\alpha)U^*=\pi_2(\zeta_\alpha)\text{ in }PU(\mathcal{H}_2) \tag{3.1.3}$$

Notice that if each π_i is the restriction of a positive energy representation of LG, any unitary equivalence $U:\mathcal{H}_1\rightarrow\mathcal{H}_2$ as LG–modules necessarily satisfies (3.1.3). Indeed, for any $g\in G$ with canonical lifts $\pi_i(g)\in U(\mathcal{H}_i)$,

$$U\pi_1(g)U^*=\chi(g)\pi_2(g) \tag{3.1.4}$$

for some $\chi(g)\in\mathbb{T}$ which defines a character of G. Since G is simple, $\chi\equiv1$ and (3.1.3) holds.

Since our goal is to reconstruct the finite energy subspaces of the level 1 positive energy representations of LG, we shall in fact be interested in projective representations of $T\times\check{T}$ on *pre*–Hilbert spaces $\mathcal{H}$ and demand accordingly that they be algebraically irreducible. We shall further assume that they are

the algebraic sum of their T–eigenspaces. To construct representations of $T\times\check{T}$ satisfying (3.1.1) and (3.1.2), we need some elementary facts about cocycles on Λ_R. Let Γ, A be groups with A abelian and recall that an A–valued *cocyle* on Γ is a map $\epsilon:\Gamma\times\Gamma\to A$ satisfying

$$\epsilon(\lambda,\mu)\epsilon(\lambda\mu,\rho)=\epsilon(\mu,\rho)\epsilon(\lambda,\mu\rho) \tag{3.1.5}$$

A *coboundary* $b:\Gamma\times\Gamma\to A$ is a map of the form

$$b(\lambda,\mu)=\frac{a(\lambda)a(\mu)}{a(\lambda\mu)} \tag{3.1.6}$$

for some function $a:\Gamma\to A$. As is readily verified, coboundaries always satisfy (3.1.5). Two cocyles ϵ,ϵ' are *cohomologous* if they differ by a coboundary b, *i.e.* if $\epsilon'=\epsilon b$. If Γ is abelian, we may attach to any cocycle ϵ a *commutator map* $\omega:\Gamma\times\Gamma\to A$ defined by $\omega(\lambda,\mu)=\epsilon(\lambda,\mu)\epsilon(\mu,\lambda)^{-1}$. ω depends only upon the cohomolgy class of ϵ and, by (3.1.5) is bilinear and skew–symmetric. The following is well–known, see for example [**FLM**, propn. 5.2.3] and [**FK**, §2.3]

Proposition 3.1.1. *If $\Gamma\cong\mathbb{Z}^r$, then*

(i) *The cocyles ϵ and ϵ' are cohomologous if, and only if they have the same commutator map.*
(ii) *For any skew–symmetric, bilinear form $\omega:\Gamma\times\Gamma\to A$ there exists a cocycle ϵ whose commutator map is ω. Moreover, ϵ may be chosen with the following normalisations*

$$\epsilon(\lambda,0)=1 \qquad \epsilon(0,\lambda)=1 \qquad \epsilon(\lambda,-\lambda)=1 \tag{3.1.7}$$

Let now ϵ be a $\mathbb{T}$–valued cocycle on Λ_R with commutator map $\omega(\alpha,\beta)=(-1)^{\langle\alpha,\beta\rangle}$. We consider the action of $T\times\check{T}$ on the group algebra $\mathbb{C}[\Lambda_R]$ and more generally on $\mathbb{C}[\nu+\Lambda_R]$, $\nu\in\Lambda_W$, given by

$$\pi(\tau)f\,(\mu)=\mu(\tau)f(\mu) \tag{3.1.8}$$

$$\pi(\zeta_\alpha)f=L_\alpha\epsilon_\alpha f \tag{3.1.9}$$

where L_α is the operator of left translation by α and ϵ_α acts as multiplication by the function $\mu\to\epsilon(\alpha,\mu-\nu)$, $\mu\in\nu+\Lambda_R$. Using the natural basis δ_μ of $\mathbb{C}[\nu+\Lambda_R]$, one readily verifies that $\pi(\tau)\pi(\zeta_\alpha)\pi(\tau)^*=\alpha(\tau)\pi(\zeta_\alpha)$. Moreover, (3.1.5) implies that

$$\pi(\zeta_\alpha)\pi(\zeta_\beta)=\epsilon(\alpha,\beta)\pi(\zeta_{\alpha+\beta}) \tag{3.1.10}$$

so that π is a projective unitary representation of $T\times\check{T}$ satisfying (3.1.1) and (3.1.2). Notice that the choice of the cocycle is immaterial since (i) of proposition 3.1.1 implies that any two choices ϵ,η satisfy $\epsilon=\eta\cdot b$ for some coboundary b of the form (3.1.6). Thus, if M_a acts on $\mathbb{C}[\nu+\Lambda_R]$ as multiplication by the function $\mu\to a(\mu-\nu)$, M_a commutes with T and satisfies

$$M_a\pi_\epsilon(\zeta_\alpha)M_a^*=a(\alpha)\pi_\eta(\zeta_\alpha) \tag{3.1.11}$$

and therefore gives the unitary equivalence of π_ϵ and π_η. We now have

Proposition 3.1.2.

(i) *For any $\nu\in\Lambda_W$, the representation of $T\times\check{T}$ on $\mathbb{C}[\nu+\Lambda_R]$ given by* (3.1.8)–(3.1.9) *is irreducible. Any operator on $\mathbb{C}[\nu+\Lambda_R]$ commuting with π is a scalar.*
(ii) *$\mathbb{C}[\nu+\Lambda_R]$ and $\mathbb{C}[\nu'+\Lambda_R]$ are unitarily equivalent iff $\nu-\nu'\in\Lambda_R$.*
(iii) *Let $\mathcal{H}$ be a pre–Hilbert space supporting a projective unitary representation π of $T\times\check{T}$ with a preferred unitary lift over T satisfying* (3.1.1) *and* (3.1.2). *If $\mathcal{H}$ is the algebraic direct sum of its T–weight spaces and is irreducible, then for some $\nu\in\Lambda_W$, $\mathcal{H}\cong\mathbb{C}[\nu+\Lambda_R]$.*

Proof. (i) The irreducibility of $\mathbb{C}[\nu+\Lambda_R]$ is clear since any non–zero submodule $\mathcal{K}$ is the direct sum of its weight spaces and therefore contains a basis vector δ_μ, $\mu\in\nu+\Lambda_R$. Thus, $\mathcal{H}=\check{T}\mathbb{C}\delta_\mu\subseteq\mathcal{K}$. Any operator S commuting with T necessarily acts as multiplication by a function f_S since $\pi(\tau)S\delta_\mu=\mu(\tau)S\delta_\mu$ and therefore $S\delta_\mu=f_S(\mu)\delta_\mu$ for some $f_S(\mu)\in\mathbb{T}$. If S commutes with $\check{T}$, f_S is invariant under translations and therefore $S\equiv f_S(\nu)\cdot 1$.
(ii) If $\nu-\nu'\in\Lambda_R$, then by (3.1.5)

$$\epsilon(\alpha,\mu-\nu')=\epsilon(\alpha,\mu-\nu)\frac{\epsilon(\alpha+\mu-\nu,\nu-\nu')}{\epsilon(\mu-\nu,\nu-\nu')} \tag{3.1.12}$$

for any $\alpha \in \Lambda_R$ and $\mu \in \nu + \Lambda_R$. It follows that the operator M acting on $\mathbb{C}[\nu+\Lambda_R]$ as multiplication by $\mu \to \epsilon(\mu-\nu, \nu-\nu')$ satisfies $M\pi_\nu(\zeta_\alpha)M^* = \pi_{\nu'}(\zeta_\alpha)$ and clearly commutes with T thus giving the unitary equivalence of π_ν and $\pi_{\nu'}$. The converse is clear.
(iii) Let $\mathcal{H} = \bigoplus_{\mu\in\Lambda_W} \mathcal{H}_\mu$ be the weight space decomposition of $\mathcal{H}$ for the action of T and $0 \neq v_\nu \in \mathcal{H}_\nu$ a vector of unit length. $\mathcal{K}_\nu = \bigoplus_{\alpha\in\Lambda_R} \mathbb{C}\cdot\pi(\zeta_\alpha)v_\nu$ is invariant under T by (3.1.1) and (3.1.2) and under $\check{T}$. By irreducibility, $\mathcal{H} = \mathcal{K}_\nu \cong \mathbb{C}[\nu+\Lambda_R]$ where the T–equivariant equivalence is given by $\pi(\zeta_\alpha)v_\nu \to \delta_{\alpha+\nu}$ and we may therefore assume that $T\times\check{T}$ is acting on $\mathbb{C}[\nu+\Lambda_R]$. Denoting the left–translation action of Λ_R on $\mathcal{H}$ by $\alpha \to L_\alpha$, we notice that, by (3.1.1) and (3.1.2) the operator $L_\alpha^*\pi(\zeta_\alpha)$ commutes with T and therefore acts as multiplication by a $\mathbb{T}$–valued function $\mu \to \eta(\alpha,\mu)$. Normalising the lifts $\pi(\zeta_\alpha) \in U(\mathcal{H})$, we may assume that $\eta(\alpha,\nu) = 1$ for any α. The $\pi(\zeta_\alpha)$ give a projective representation of $\check{T}$ and therefore satisfy

$$\pi(\zeta_\alpha)\pi(\zeta_\beta) = \epsilon(\alpha,\beta)\pi(\zeta_{\alpha+\beta}) \tag{3.1.13}$$

for some function $\epsilon : \Lambda_R \times \Lambda_R \to \mathbb{T}$ which, by the associativity of the multiplication in $U(\mathcal{H})$ and (3.1.1) and (3.1.2) is a cocycle with commutator map ω. Since $\pi(\zeta_\alpha) = L_\alpha\eta(\alpha,\cdot)$, (3.1.13) yields, for any $\mu \in \nu+\Lambda_R$

$$\eta(\alpha,\beta+\mu)\eta(\beta,\mu) = \epsilon(\alpha,\beta)\eta(\alpha+\beta,\mu) \tag{3.1.14}$$

Evaluating at $\mu = \nu$ yields $\eta(\alpha,\mu) = \epsilon(\alpha,\mu-\nu)$ $\diamond$

Recall now from section 2 that, on the finite energy subspace of a positive energy representation, the infinitesimal generator of rotations d and the generating function $X_\alpha(z)$ satisfy

$$[d,\pi(\tau)] = 0 \qquad\qquad [d,\pi(\zeta_\alpha)] = \pi(\zeta_\alpha)(\pi(\alpha) + \frac{\langle\alpha,\alpha\rangle}{2}) \tag{3.1.15}$$

$$\pi(\tau)X_\alpha(z)\pi(\tau)^* = \alpha(\tau)X_\alpha \qquad\qquad \pi(\zeta_\beta)X_\alpha(z)\pi(\zeta_\beta)^* = z^{-\langle\beta,\alpha\rangle}X_\alpha(z) \tag{3.1.16}$$

We now give an explicit construction of operators possessing the same commutation relations in any of the irreducible $T\times\check{T}$–modules constructed above.

Proposition 3.1.3. *Let $\mathcal{H} = \mathbb{C}[\nu+\Lambda_R]$ be the irreducible representation of $T\times\check{T}$ given by* (3.1.8)–(3.1.9)*. Then*

(i) *Any operator d on $\mathcal{H}$ satisfying* (3.1.15) *is given, up to addition by a constant, by*

$$d = \frac{1}{2}\sum_k \pi(h_k)\pi(h^k) \tag{3.1.17}$$

where h_k, h^k are dual basis of $\mathfrak{t}_\mathbb{C}$ with respect to the basic inner product.

(ii) *For $\alpha \in \Lambda_R$, the unique solution in $X_\alpha(z) \in \mathrm{End}(\mathcal{H})[[z^{-1},z]]$ to the commutation relations* (3.1.16) *and*

$$[d, X_\alpha(z)] = z\frac{d}{dz}X_\alpha(z) \tag{3.1.18}$$

is given, up to a multiplicative constant by

$$X_\alpha(z) = V_\alpha z^{\alpha+\frac{\langle\alpha,\alpha\rangle}{2}} \tag{3.1.19}$$

where $V_\alpha = L_\alpha\epsilon_\alpha^\dagger$ and $\epsilon_\alpha^\dagger$ acts as multiplication by the function $\mu \to \epsilon(\mu-\nu,\alpha)$ so that

$$V_\alpha V_\beta = \epsilon(\beta,\alpha)V_{\alpha+\beta} \tag{3.1.20}$$

$$V_\alpha z^\beta = z^{\beta-\langle\alpha,\beta\rangle}V_\alpha \tag{3.1.21}$$

(iii) *If the cocycle ϵ giving the action* (3.1.9) *is chosen with the normalisations* (3.1.7)*, then $X_0(z) = 1$ and $X_\alpha(z)$ has the formal adjunction property $X_\alpha(z)^* = X_{-\alpha}(z)$.*

Proof. (i) The infinitesimal action of $\mathfrak{t}_\mathbb{C}$ on $\mathcal{H}$ corresponding to π is given by $\pi(h)\delta_\mu = \langle h,\mu\rangle\delta_\mu$ and therefore satisfies

$$[h,\pi(\zeta_\alpha)] = \langle h,\alpha\rangle\pi(\zeta_\alpha) \tag{3.1.22}$$

It follows that the operator d given by (3.1.17) satisfies (3.1.15). The uniqueness of d follows from (i) of proposition 3.1.2.

(ii) By (3.1.22), the operator z^α satisfies

$$\pi(\zeta_\beta)z^\alpha\pi(\zeta_\beta)^* = z^{\alpha-\langle\alpha,\beta\rangle} \tag{3.1.23}$$

As is readily verified,

$$\pi(\tau)V_\alpha\pi(\tau)^* = \alpha(\tau)V_\alpha \qquad \text{and} \qquad \pi(\zeta_\beta)V_\alpha\pi(\zeta_\beta)^* = V_\alpha \tag{3.1.24}$$

and it follows that $X_\alpha(z) = V_\alpha z^{\alpha+\frac{\langle\alpha,\alpha\rangle}{2}}$ satisfies (3.1.16). Moreover, using (3.1.15)

$$[d, X_\alpha(z)] = z\frac{d}{dz}X_\alpha(z) \tag{3.1.25}$$

so that $X_\alpha(z)$ does indeed satisfy the required commutation relations. The relations (3.1.20) and (3.1.21) follow at once from (3.1.5) and the fact that $L_\alpha\beta L_\alpha^* = \beta - \langle\alpha,\beta\rangle$ respectively. The uniqueness of the solution follows from irreducibility. Indeed, if $Y_\alpha(z)$ satisfies the required commutation relations, then $Z_\alpha(z) = Y_\alpha(z)X_{-\alpha}(z)$ commutes with $T\times\check{T}$ and must therefore be a scalar function of z. (3.1.18) then implies that $Z_\alpha(z)$ is a scalar.
(iii) If (3.1.7) hold, then $V_0 = 1$ and $V_\alpha V_{-\alpha} = \epsilon(-\alpha,\alpha)V_0 = 1$ hence $V_\alpha^* = V_{-\alpha}$ from which the claimed adjunction property follows. ◇

Remark. Notice that the operator defined by (3.1.19) only involves integral powers of z since Λ_R is even and $\langle\alpha,\mu\rangle \in \mathbb{Z}$ for any $\alpha\in\Lambda_R$ and $\mu\in\Lambda_W$.

We conclude this subsection by addressing a minor technical point, namely the uniqueness of the direct sum of projective representations of $T\times\check{T}$ satisfying (3.1.1)–(3.1.2). Recall from §1.3 of chapter I that a direct sum of projective representations $(\pi_i,\mathcal{H}_i)$ of a group Γ is a projective representation of Γ on $\bigoplus\mathcal{H}_i$ restricting on each $\mathcal{H}_i$ to π_i. One such exists iff the pull–backs of the central extensions

$$1\to\mathbb{T}\to U(\mathcal{H}_i)\to PU(\mathcal{H}_i)\to 1 \tag{3.1.26}$$

to Γ are isomorphic and depends upon the choice of identifications $\pi_i^*U(\mathcal{H}_i)\cong\pi_j^*U(\mathcal{H}_j)$. Since these are unique only up to multiplication by a character of Γ, a canonically defined direct sum does not exist in general. When $\Gamma = T\times\check{T}$, the following holds however

Proposition 3.1.4. *Let $(\pi_i,\mathcal{H}_i)$ be projective unitary representations of $T\times\check{T}$ satisfying* (3.1.1) *and* (3.1.2). *Then, there exists a direct sum representation of $T\times\check{T}$ on $\bigoplus_i\mathcal{H}_i$, unique up to unitary equivalence.*

Proof. Let ϵ be a cocycle on Λ_R with associated commutator map ω and choose lifts $U_\alpha^i\in U(\mathcal{H}_i)$ of $\pi_i(\zeta_\alpha)$ satisfying $U_\alpha^iU_\beta^i = \epsilon(\alpha,\beta)U_{\alpha+\beta}^i$. The representation of $T\times\check{T}$ on $\bigoplus\mathcal{H}_i$ obtained by letting $\tau\in\mathbb{T}$ act as $\bigoplus_i\pi_i(\tau)$ and ζ_α as $\bigoplus_i U_\alpha^i$ clearly is a direct sum representation of the $\mathcal{H}_i$. Let now π,ρ be two representations on $\mathcal{H} = \bigoplus\mathcal{H}_i$ leaving each $\mathcal{H}_i$ restricting on each $\mathcal{H}_i$ to unitarily equivalent representations. Let ϵ be a cocycle with commutator map ω and choose lifts $U_\alpha, V_\alpha\in U(\mathcal{H})$ of $\pi(\zeta_\alpha),\rho(\zeta_\alpha)$ respectively satisfying

$$U_\alpha U_\beta = \epsilon(\alpha,\beta)U_{\alpha+\beta} \qquad \text{and} \qquad V_\alpha V_\beta = \epsilon(\alpha,\beta)V_{\alpha+\beta} \tag{3.1.27}$$

By assumption, there exist unitaries $I_i:\mathcal{H}_i\to\mathcal{H}_i$ such that

$$I_iP_i\pi(\tau)P_iI_i^* = P_i\rho(\tau)P_i \qquad \text{and} \qquad I_iP_iU_\alpha P_iI_i^* = \chi_i(\alpha)P_iV_\alpha P_i \tag{3.1.28}$$

where P_i is the orthogonal projection onto $\mathcal{H}_i$ and $\chi_i(\alpha)\in\mathbb{T}$. Comparing (3.1.27) and (3.1.28) shows that each χ_i is a character of Λ_R and is therefore given by $\alpha\to\alpha(\tau_i)$ for some $\tau_i\in T$. It follows from (3.1.1) and (3.1.2) that

$$\pi(\tau_i)I_iP_iU_\alpha P_iI_i^*\pi(\tau_i)^* = P_iV_\alpha P_i \tag{3.1.29}$$

Since $\pi(\tau_i)I_i$ intertwines the action of T, the unitary $\bigoplus_i\pi(\tau_i)I_i$ gives the equivalence of π and ρ ◇

3.2. Stone–von Neumann theorem and vertex operators for $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}$.

This subsection follows [**FLM**, chap. 3]. Consider the action of $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$ on the finite energy subspace of a level 1 positive energy representation of LG. It satisfies

$$[h_1(n), h_2(m)] = n\delta_{n+m,0}\langle h_1, h_2\rangle \tag{3.2.1}$$

$$[d, h(n)] = -nh(n) \tag{3.2.2}$$

as well as the formal adjunction property

$$h(n)^* = -\overline{h}(-n) \tag{3.2.3}$$

where $\overline{h}$ is the canonical conjugation on $\mathfrak{t}_{\mathbb{C}}$. An explicit representation of $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$ satisfying the above may be obtained as follows. Write $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}} = V_+ \bigoplus V_-$ where the $V_\pm$ are spanned by the $h(n) = h \otimes e^{in\theta}$ with $n \lessgtr 0$. Define an action of $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}} \rtimes \mathbb{C}d$ on the symmetric algebra

$$\mathcal{S} = SV_+ = \bigoplus_k S^k V_+ \tag{3.2.4}$$

by

$$h(-n)\, h_1(-n_1) \otimes \cdots \otimes h_k(-n_k) = h(-n) \otimes h_1(-n_1) \otimes \cdots \otimes h_k(-n_k) \tag{3.2.5}$$

$$h(n)\, h_1(-n_1) \otimes \cdots \otimes h_k(-n_k) = \sum_j n\delta_{n,n_j}\langle h, h_j\rangle h_1(-n_1) \otimes \cdots \otimes \widehat{h_j(-n_j)} \otimes \cdots \otimes h_k(-n_k) \tag{3.2.6}$$

$$d\, h_1(-n_1) \otimes \cdots \otimes h_k(-n_k) = (n_1 + \cdots + n_k) h_1(-n_1) \otimes \cdots \otimes h_k(-n_k) \tag{3.2.7}$$

where $n, n_1, \dots n_k > 0$ throughout. It is readily verified that the relations (3.2.1)–(3.2.2) hold. Moreover, the hermitian form $(\cdot,\cdot)$ defined on V_+ by $(h_1(n), h_2(m)) = n\delta_{n,m}\langle h_1, \overline{h_2}\rangle$ is positive definite since $h_1 \otimes h_2 \to -\langle h_1, \overline{h_2}\rangle$ is a positive definite inner product on $\mathfrak{t}_{\mathbb{C}}$. $(\cdot,\cdot)$ yields an inner product on $\mathcal{S}$ for which (3.2.3) holds.

Let $\mathcal{H}$ be a pre–Hilbert space supporting a representation of $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$ satisfying (3.2.1) and (3.2.3). By definition, $\mathcal{H}$ is of positive energy if $\mathcal{H}$ has an $\mathbb{N}$–grading $\mathcal{H} = \bigoplus_{n\geq 0}\mathcal{H}(n)$, with $\dim H(n) < \infty$ and the operator d acting as multiplication by n on $\mathcal{H}(n)$ satisfies (3.2.2).

Proposition 3.2.1.

(i) *The following holds on $\mathcal{S}$,*

$$d = \sum_{m>0} h_k(-m)h^k(m) \tag{3.2.8}$$

where h_k, h^k are dual basis of $\mathfrak{t}_{\mathbb{C}}$ for the basic inner product $\langle\cdot,\cdot\rangle$.

(ii) *$\mathcal{S} = SV_+$ is algebraically irreducible under the action of $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$. Moreover, $T \in \mathrm{End}(\mathcal{S})$ commutes with the $h(n)$ iff T is a scalar.*

(iii) *Let $\mathcal{H}$ be a unitary, positive energy representation of $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$ and define*

$$\mathcal{V}_{\mathcal{H}} = \{\nu \in \mathcal{H} |\, h(n)\nu = 0 \text{ for any } h \in \mathfrak{t}_{\mathbb{C}} \text{ and } n > 0\} \tag{3.2.9}$$

Then $\mathcal{H} \cong \mathcal{S} \otimes \mathcal{V}_{\mathcal{H}}$ where the unitary equivalence is given by $p \cdot 1 \otimes \nu \to p\nu$ where p is any polynomial in the $h(n)$ and $\nu \in \mathcal{V}_{\mathcal{H}}$.

Proof. (i) Let D be the operator given by the right hand–side of (3.2.8) and notice that D is well–defined since the $h(m)$, $m > 0$ are locally nilpotent on $\mathcal{S}$. It follows from (3.2.1) that

$$[D, h(n)] = -nh(n) = [d, h(n)] \tag{3.2.10}$$

Denoting the lowest energy vector in $\mathcal{S}$ by $1 \in S^0V_+$, we have $D1 = 0 = d1$ and it follows by cyclicity of 1 under the action of $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$ that $D = d$.
(ii) If $\mathcal{V} \subset \mathcal{S}$ is invariant under the $h(n)$, it is invariant under d by (3.2.8). Consequently, it contains a non–zero lowest energy vector ν necessarily annihilated by the energy decreasing $h(n)$, $n > 0$. This can only be if $\nu \in S^0V_+$ and then $\mathcal{S} = \mathfrak{U}(L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}})S^0V_+ \subset \mathcal{V}$, where $\mathfrak{U}$ denotes the enveloping algebra.

Similarly, any T commuting with the $h(n)$ commutes with d and consequently leaves $S^0 V_+$ invariant. It therefore acts on it as multiplication by a scalar λ and by cyclicity, $T \equiv \lambda$.
(iii) Let $\nu \in \mathcal{V}_{\mathcal{H}}$. We claim that the map $\mathcal{S} \to \mathcal{H}$ given by $p \cdot 1 \to p\nu$ is norm–preserving and therefore injective. Let p, q be polynomials in $\mathfrak{U}(L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}})$ so that $(p\nu, q\nu) = (q^* p\nu, \nu)$. Using the commutation relations of the $h(n)$'s, the product $q^* p$ may be written as the sum of a constant term τ_0 and terms of the form $h_1(-n_1) \cdot h_k(-n_k) h_1'(m_1) \cdot h_l'(m_l)$ where the n_i and m_j are positive. However,

$$(h_1(-n_1) \cdots h_k(-n_k) h_1'(m_1) \cdots h_l'(m_l)\nu, \nu) = (-1)^k (h_1'(m_1) \cdots h_l'(m_l)\nu, \overline{h}_k(n_k) \cdots \overline{h}_1(n_1)\nu) = 0 \tag{3.2.11}$$

and therefore $(p\nu, q\nu) = \tau_0 \|\nu\|^2$. Similarly, $(p \cdot 1, q \cdot 1) = \tau_0$ and it follows that $p \cdot 1 \otimes \nu \to p\nu$ is an $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$–equivariant isometry $\mathcal{S} \otimes \mathcal{V}_{\mathcal{H}} \to \mathcal{H}$. To see that it is surjective, notice that $\mathcal{V}_{\mathcal{H}}$ is invariant under d so that $\mathcal{S} \otimes \mathcal{V}_{\mathcal{H}}$ is a graded submodule of $\mathcal{H}$. Since the eigenspaces of d are finite–dimensional, it possesses a graded orthogonal complement $\mathcal{K}$ invariant under $L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$. If $\mathcal{K} \neq \{0\}$, it possesses a lowest energy vector ξ since d is bounded below. However, ξ is necessarily annihilated by the $h(n)$, $n > 0$ and therefore lies in $\mathcal{V}_{\mathcal{H}}$, a contradiction $\diamond$

Using $\langle \cdot, \cdot \rangle$ to identify $\mathfrak{t}_{\mathbb{C}}^*$ with $\mathfrak{t}_{\mathbb{C}}$, we consider, for any $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ the elements $\alpha(m) \in L^{\mathrm{pol}}\mathfrak{t}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$ and introduce the exponential operators

$$E^{\pm}(\alpha, z) = \exp\Big(-\sum_{m \gtrless 0} \frac{\alpha(m)}{m} z^{-m}\Big) \tag{3.2.12}$$

These are to be viewed as formal Laurent series with coefficents in the endomorphisms of $\mathcal{S}$. We will need the following elementary

LEMMA 3.2.2. *If $[a, b]$ commutes with both a and b, then $[a, e^b] = [a, b]e^b$ and $e^a e^b = e^{[a,b]} e^b e^a$.*

PROOF. Induction shows that $[a, b^n] = n[a, b]b^{n-1}$ and therefore $[a, e^b] = [a, b]e^b$. It follows that $ae^b = e^b(a + [a, b])$ thus $a^n e^b = e^b(a + [a, b])^n$ whence $e^a e^b = e^{[a,b]} e^b e^a$ $\diamond$

PROPOSITION 3.2.3. *The following commutation relations hold*

$$E^{\pm}(\alpha, z)E^{\pm}(\beta, z) = E^{\pm}(\alpha + \beta, z) \tag{3.2.13}$$

$$E^{\pm}(\alpha, z)E^{\pm}(-\alpha, z) = 1 \tag{3.2.14}$$

$$E^{\pm}(\alpha, z)^* = E^{\mp}(\overline{\alpha}, z) \tag{3.2.15}$$

$$E^{\pm}(\alpha, z)E^{\pm}(\beta, \zeta) = E^{\pm}(\beta, \zeta)E^{\pm}(\alpha, z) \tag{3.2.16}$$

$$E^{+}(\alpha, z)E^{-}(\beta, \zeta) = \Big(1 - \frac{\zeta}{z}\Big)^{\langle \alpha, \beta \rangle} E^{-}(\beta, \zeta)E^{+}(\alpha, z) \tag{3.2.17}$$

where $\Big(1 - \frac{\zeta}{z}\Big)^{\langle \alpha, \beta \rangle}$ *is defined by* (1.7).

PROOF. All but the last relation are trivial. The last follows from lemma 3.2.2 and the fact that

$$[\sum_{m>0} \frac{\alpha(m)}{m} z^{-m}, \sum_{n<0} \frac{\beta(n)}{n} \zeta^{-n}] = -\langle \alpha, \beta \rangle \sum_{m>0} \frac{1}{m} \Big(\frac{\zeta}{z}\Big)^m = \langle \alpha, \beta \rangle \log\Big(1 - \frac{\zeta}{z}\Big) \tag{3.2.18}$$

$\diamond$

Proposition 3.2.4.

(i) *For any $\alpha \in \mathfrak{t}_{\mathbb{C}}$, the formal Laurent series $X_\alpha(z) = E^-(\alpha, z)E^+(\alpha, z) \in \mathrm{End}(\mathcal{S})[[z, z^{-1}]]$ is well–defined, satisfies*

$$[h(n), X_\alpha(z)] = \langle h, \alpha\rangle z^n X_\alpha(z) \tag{3.2.19}$$

$$[d, X_\alpha(z)] = z\frac{d}{dz}X_\alpha(z) \tag{3.2.20}$$

and the formal adjunction property $X_\alpha(z)^* = X_{\overline{\alpha}}(z)$.

(ii) *The above commutation relations characterise $X_\alpha(z)$ uniquely, up to a multiplicative constant.*

Proof. (i) $X_\alpha(z)$ is well–defined on $\mathcal{S}$ since, for any $\psi \in \mathcal{S}$, only finitely many terms in $E^+(\alpha, z)\psi$ do not vanish. Moreover, $[h(n), -\sum_{m\gtrless 0} \frac{\alpha(m)}{m} z^{-m}] = z^n\langle h, \alpha\rangle\delta_{n\lessgtr 0}$ and therefore, by lemma 3.2.2, $[h(n), X_\alpha(z)] = z^n\langle h, \alpha\rangle X_\alpha(z)$. The second relation follows from the general fact that if $\phi(z) = \sum_n \phi_n z^{-n}$ satisfies $[d, \phi(z)] = z\frac{d}{dz}\phi(z)$ or equivalently $[d, \phi_n] = -n\phi_n$, then so do $p(\phi(z))$ and $\exp(\phi(z))$ for any polynomial p, whenever these are well–defined. The adjunction property of $X_\alpha(z)$ follows from those of the $E^\pm(\alpha, z)$.
(ii) Let $Y_\alpha \in \mathrm{End}(\mathcal{S})[[z, z^{-1}]]$ satisfy (3.2.19)–(3.2.20). The operator $\phi(z) = E^-(-\alpha, z)Y_\alpha(z)E^+(-\alpha, z)$ is easily seen to be a well–defined element of $\mathrm{End}(\mathcal{S})[[z, z^{-1}]]$ commuting with the $h(n)$. By (i), it follows that $\phi(z) = \sum_n a_n z^{-n}$ for some $a_n \in \mathbb{C}$. The relation (3.2.20) however implies that $a_n = 0$ if $n \neq 0$ ◇

3.3. Irreducible representations of $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t} \times T \times \check{T}$.

Proposition 3.3.1.

(i) *The tensor product representations of $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}\times T\times\check{T}$ on the modules $\mathcal{S}\otimes\mathbb{C}[\nu+\Lambda_R]$ given by* (3.1.8)–(3.1.9) *and* (3.2.5)–(3.2.6) *are irreducible. Moreover, $\mathcal{S}\otimes\mathbb{C}[\nu+\Lambda_R]$ and $\mathcal{S}\otimes\mathbb{C}[\nu'+\Lambda_R]$ are equivalent if, and only if $\nu - \nu' \in \Lambda_R$.*

(ii) *Let $\mathcal{H}$ be a unitary positive energy representation of $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}\times T\times\check{T}$ satisfying* (3.1.1)–(3.1.2), (3.2.1) *and* (3.1)–(3.2). *If $\mathcal{H}$ is irreducible, it is unitarily equivalent to $\mathcal{S}\otimes\mathbb{C}[\nu+\Lambda_R]$ for some $\nu \in \Lambda_W$.*

Proof. (i) Let $0 \neq V \subset \mathcal{S}\otimes\mathbb{C}[\nu+\Lambda_R]$ be a submodule. V is the direct sum of its T–weight spaces which are necessarily of the form $V_\mu \otimes \delta_\mu$ where $V_\mu \subset \mathcal{S}$ is invariant under $L\mathfrak{t}/\mathfrak{t}$ and therefore equal to 0 or $\mathcal{S}$ by proposition 3.2.1. Thus, $V_\mu = \mathcal{S}$ for some μ and therefore, using the action of $\check{T}$, $V_\mu = \mathcal{S}$ for all $\mu \in \nu + \Lambda_R$ whence $V = \mathcal{S}$. The last statement follows from proposition 3.1.2.
(ii) By proposition 3.2.1, $\mathcal{H} \cong \mathcal{S}\otimes\mathcal{V}_{\mathcal{H}}$. Clearly, $T\times\check{T}$ leaves $\mathcal{V}_{\mathcal{H}}$ invariant and the isomorphism is easily seen to be equivariant for this action. By irreducibility of $\mathcal{H}$, $\mathcal{V}_{\mathcal{H}}$ is irreducible under $T\times\check{T}$ and therefore, by proposition 3.1.2 of the form $\mathbb{C}[\nu+\Lambda_R]$ for some $\nu \in \Lambda_W$ ◇

4. The vertex operator construction

This section follows [**FLM**, chap. 7] and [**GO2**, §6]. We show below that the vertex operators $X_\alpha(z)$ give rise, in any of the irreducible representations of $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}\times T\times\check{T}$ classified in section 3, to level 1 irreducible positive energy representations of $L^{\mathrm{pol}}\mathfrak{g}$ and that the latter may all be obtained in this way.

It will in fact be more convenient to work with the direct sum

$$\mathcal{S}\otimes\mathbb{C}[\Lambda_W] = \bigoplus_{\nu\in\Lambda_W/\Lambda_R} \mathcal{S}\otimes\mathbb{C}[\nu+\Lambda_R] \tag{4.1}$$

of all irreducible $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}\times T\times\check{T}$–modules. By proposition 3.1.4, the action of $T\times\check{T}$ on the factor $\mathbb{C}[\Lambda_W]$ is unambiguously defined and we may, by uniqueness assume that it extends to one of $T\times\Lambda_W$ given by

$$\pi(\tau)f(\mu) = \mu(\tau)f(\mu) \tag{4.2}$$

$$\pi(\lambda)f = L_\lambda\epsilon_\lambda f \tag{4.3}$$

As customary, L_λ is the left–translation operator and ϵ_λ acts as multiplication by the function $\mu \to \epsilon(\lambda,\mu)$ where ϵ is a $\mathbb{T}$–valued cocycle on Λ_W whose commutator map $\widetilde{\omega}$ satisfies $\widetilde{\omega}(\alpha,\beta) = (-1)^{\langle\alpha,\beta\rangle}$ whenever $\alpha,\beta \in \Lambda_R$. The choice of a particular $\widetilde{\omega}$ satisfying this requirement is clearly immaterial.

Choose ϵ to be normalised in the sense of (3.1.7) and let $\epsilon^\dagger(\lambda,\mu) = \epsilon(\mu,\lambda)$ be the transposed cocycle. Let

$$U_\lambda := \pi(\lambda) = L_\lambda \epsilon_\lambda \tag{4.4}$$

and $V_\lambda = L_\lambda \epsilon_\lambda^\dagger$ so that

$$U_\lambda U_\mu = \epsilon(\lambda,\mu) U_{\lambda+\mu} \tag{4.5}$$
$$V_\lambda V_\mu = \epsilon(\mu,\lambda) V_{\lambda+\mu} \tag{4.6}$$
$$U_\lambda V_\mu U_\lambda^* V_\mu^* = 1 \tag{4.7}$$

and define for any $\alpha \in \Lambda_R$, the vertex operators

$$X_\alpha(z) := E^-(\alpha,z) V_\alpha z^{\alpha + \frac{\langle\alpha,\alpha\rangle}{2}} E^+(\alpha,z) = \sum_{n\in\mathbb{Z}} X_\alpha(n) z^{-n} \tag{4.8}$$

We shall need the following

Lemma 4.1. *Let $\alpha \in \Lambda_R$. Then $\langle\alpha,\alpha\rangle = 2$ iff α is a root.*

Proof. Let $\alpha \in \Lambda_R$ of length $\sqrt{2}$. It is clearly sufficient to show that $w\alpha$ is a root for some $w \in W$. Write $\alpha = \beta_1 + \cdots + \beta_r$ where the β_i are roots. Clearly, one cannot have $\langle\alpha,\beta_i\rangle \le 0$ for all i else

$$\|\alpha - \sum \beta_i\|^2 = \|\alpha\|^2 - 2\sum\langle\alpha,\beta_i\rangle + \|\sum\beta_i\|^2 > 0 \tag{4.9}$$

Thus there exists a β_{i_1} with $\langle\alpha,\beta_{i_1}\rangle \in \{1,2\}$. If $\langle\alpha,\beta_{i_1}\rangle = 2$ then $\alpha = \beta_{i_1}$ and α is a root. Otherwise, let $\sigma_{\beta_{i_1}} \in W$ be the simple reflection corresponding to β_{i_1}, then

$$\sigma_{\beta_{i_1}}\alpha = \alpha - \beta_{i_1} = \sum_{i\neq i_1}\beta_i \tag{4.10}$$

Iterating this argument shows that $\sigma_{\beta_{i_k}} \cdots \sigma_{\beta_{i_1}}\alpha$ is a root for some sequence of distinct $1 \le i_j \le r$ $\diamond$

Theorem 4.2. *There exists a basis of $\mathfrak{g}_\mathbb{C}/\mathfrak{t}_\mathbb{C}$ given by root vectors x_α such that*

$$\pi(x_\alpha(n)) = X_\alpha(n) \tag{4.11}$$
$$\pi(h(n)) = h(n) \; n \neq 0 \tag{4.12}$$
$$\pi(h) = h \tag{4.13}$$
$$\pi(d) = \frac{1}{2} h_k h^k + \sum_{m>0} h_k(-m) h^k(m) \tag{4.14}$$

defines a level 1 positive energy representation of $L^{\mathrm{pol}}\mathfrak{g}_\mathbb{C} \rtimes \mathbb{C}d$ on $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$.

Proof. We begin by computing the commutation relations of the vertex operators $X_\alpha(z)$. Using (3.2.17), (4.6) and $V_\beta^* z^\alpha V_\beta = z^{\alpha+\langle\alpha,\beta\rangle}$, we get

$$X_\alpha(z) X_\beta(\zeta) = \epsilon(\beta,\alpha)\Big(1 - \frac{\zeta}{z}\Big)^{\langle\alpha,\beta\rangle}\Big(\frac{z}{\zeta}\Big)^{\frac{\langle\alpha,\beta\rangle}{2}} V_{\alpha+\beta} z^{\alpha + \frac{\langle\alpha,\alpha\rangle}{2} + \frac{\langle\alpha,\beta\rangle}{2}} \zeta^{\beta + \frac{\langle\beta,\beta\rangle}{2} + \frac{\langle\alpha,\beta\rangle}{2}} \mathcal{R}(\alpha,\beta,z,\zeta) \tag{4.15}$$

where

$$\mathcal{R}(\alpha,\beta,z,\zeta) = E^-(\alpha,z)E^-(\beta,\zeta)E^+(\alpha,z)E^+(\beta,\zeta) = \mathcal{R}(\beta,\alpha,\zeta,z) \tag{4.16}$$

Similarly,

$$X_\beta(\zeta) X_\alpha(z) = \epsilon(\alpha,\beta)\Big(1 - \frac{z}{\zeta}\Big)^{\langle\alpha,\beta\rangle}\Big(\frac{\zeta}{z}\Big)^{\frac{\langle\alpha,\beta\rangle}{2}} V_{\alpha+\beta} z^{\alpha + \frac{\langle\alpha,\alpha\rangle}{2} + \frac{\langle\alpha,\beta\rangle}{2}} \zeta^{\beta + \frac{\langle\beta,\beta\rangle}{2} + \frac{\langle\alpha,\beta\rangle}{2}} \mathcal{R}(\alpha,\beta,z,\zeta) \tag{4.17}$$

Since $\epsilon(\alpha,\beta) = (-1)^{\langle\alpha,\beta\rangle}\epsilon(\beta,\alpha)$, we get

$$[X_\alpha(z), X_\beta(\zeta)] = \epsilon(\beta,\alpha)\mathcal{D}(z,\zeta) V_{\alpha+\beta} z^{\alpha + \frac{\langle\alpha,\alpha\rangle}{2} + \frac{\langle\alpha,\beta\rangle}{2}} \zeta^{\beta + \frac{\langle\beta,\beta\rangle}{2} + \frac{\langle\alpha,\beta\rangle}{2}} \mathcal{R}(\alpha,\beta,z,\zeta) \tag{4.18}$$

where, using (1.8)–(1.9)

$$\mathcal{D}(z,\zeta) = \left(\frac{\zeta}{z}\right)^{-\frac{\langle\alpha,\beta\rangle}{2}} \left[\left(1-\frac{\zeta}{z}\right)^{\langle\alpha,\beta\rangle} - (-1)^{\langle\alpha,\beta\rangle}\left(1-\frac{z}{\zeta}\right)^{\langle\alpha,\beta\rangle}\left(\frac{z}{\zeta}\right)^{-\langle\alpha,\beta\rangle}\right]$$
$$= \begin{cases} \delta'\left(\frac{\zeta}{z}\right) & \text{if } \langle\alpha,\beta\rangle = -2 \\ \left(\frac{\zeta}{z}\right)^{\frac{1}{2}}\delta\left(\frac{\zeta}{z}\right) & \text{if } \langle\alpha,\beta\rangle = -1 \\ 0 & \text{if } \langle\alpha,\beta\rangle \geq 0 \end{cases} \tag{4.19}$$

Let $\Delta \subset \Lambda_R$ be the set of elements of squared length 2. Then, for $\alpha, \beta \in \Delta$, we get by (1.10)–(1.11) and (3.1.7),

$$[X_\alpha(z), X_\beta(\zeta)] = \begin{cases} \alpha(\zeta)\delta\left(\frac{\zeta}{z}\right) + \delta'\left(\frac{\zeta}{z}\right) & \text{if } \alpha+\beta = 0 & \textit{i.e.} \text{ iff } \langle\alpha,\beta\rangle = -2 \\ \epsilon(\beta,\alpha)X_{\alpha+\beta}(\zeta)\delta\left(\frac{\zeta}{z}\right) & \text{if } \alpha+\beta \in \Delta & \textit{i.e.} \text{ iff } \langle\alpha,\beta\rangle = -1 \\ 0 & \text{if } \alpha+\beta \notin \Delta\cup\{0\} & \textit{i.e.} \text{ iff } \langle\alpha,\beta\rangle \geq 0 \end{cases} \tag{4.20}$$

The commutation relations (4.20) show that the complex vector space $\mathfrak{l} \subset \mathrm{End}(\mathcal{S}\otimes\mathbb{C}[\Lambda_W])$ spanned by the operators $X_\alpha(0)$ and h, with $\alpha \in \Delta$ and $h \in \mathfrak{t}_\mathbb{C}$ is closed under the commutator bracket and is a simple Lie algebra. By lemma 4.1, $\mathfrak{l}$ and $\mathfrak{g}_\mathbb{C}$ have the same root system and it follows that there exists a Lie algebra isomorphism $\pi : \mathfrak{g}_\mathbb{C} \to \mathfrak{l}$ acting as the identity on the Cartan subalgebra $\mathfrak{t}_\mathbb{C}$. Set $x_\alpha = \pi^{-1}(X_\alpha(0))$. The Lie algebra spanned by the modes of the $X_\alpha(z)$ and the $h(n)$ is clearly a central extension of $\mathfrak{l}[z,z^{-1}] \cong \mathfrak{g}_\mathbb{C}[z,z^{-1}]$. By (4.20), the corresponding cocycle is given by $a(m)\otimes b(n) \to m\delta_{m+n,0}(a,b)$ where $(\cdot,\cdot)$ is an ad–invariant bilinear form on $\mathfrak{l}$ and is therefore a multiple of the basic inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_\mathbb{C} \cong \mathfrak{l}$. By construction however, the two coincide on $\mathfrak{t}_\mathbb{C}$ and therefore on $\mathfrak{g}_\mathbb{C}$ and it follows that $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ is a level 1 representation of $L^{\mathrm{pol}}\mathfrak{g}_\mathbb{C}$. To conclude, notice that the operator d given by (4.14) satisfies by construction $[d, X_\alpha(n)] = -nX_\alpha(n)$ and $[d, h(m)] = -mh(m)$, is bounded below and has finite–dimensional eigenspaces. Finally, the unitarity of the representation follows from the formal adjunction property $h(z)^* = -\overline{h}(z)$ and $X_\alpha(z)^* = X_{-\alpha}(z)$ $\diamond$

THEOREM 4.3. *Each $\mathcal{S}\otimes\mathbb{C}[\nu+\Lambda_R] \subset \mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ is invariant and irreducible under $L^{\mathrm{pol}}\mathfrak{g}$ and is the level 1 positive energy representation whose lowest energy subspace is the minimal G–module with weights lying in $\nu+\Lambda_R$. The corresponding highest weight vector is $1\otimes\delta_\mu$ where $\mu \in \nu+\Lambda_R$ is the unique minimal dominant weight. It follows that $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ is the direct sum of all level 1 representations of $L^{\mathrm{pol}}\mathfrak{g}$, each with multiplicity one.*

PROOF. We follow [**GO2**, §6.4]. By construction, $\mathcal{S}\otimes\mathbb{C}[\nu+\Lambda_R]$ is invariant under $L^{\mathrm{pol}}\mathfrak{t}/\mathfrak{t}\times T\times\check{T}$ and therefore under the $X_\alpha(z)$. Since $\mathcal{S}\otimes\mathbb{C}[\nu+\Lambda_R]$ is a unitary representation, it is the sum of its irreducible summands. These correspond exactly to and are generated by highest weight vectors, *i.e.* elements Ω_μ satisfying, for some $\mu\in\nu+\Lambda_R$

$$h(0)\Omega_\mu = \langle h,\mu\rangle\Omega_\mu \quad h\in\mathfrak{t}_\mathbb{C} \tag{4.21}$$

$$h(n)\Omega_\mu = 0 \quad h\in\mathfrak{t}_\mathbb{C},\ n>0 \tag{4.22}$$

$$X_\alpha(n)\Omega_\mu = 0 \quad n>0 \text{ or } n=0,\ \alpha>0 \tag{4.23}$$

The second condition implies that $\Omega_\mu \in 1\otimes\mathbb{C}[\Lambda_W]$ since this is precisely the vacuum subspace for the $h(n)$, $n>0$ and together with the first is equivalent to $\Omega_\mu = 1\otimes\delta_\mu$. Next,

$$X_\alpha(n) = \frac{1}{2\pi i}\oint\frac{dz}{z}z^n X_\alpha(z) \tag{4.24}$$

where the formal contour integration is shorthand for taking the coefficent of z^{-1} in the power series expansion of the integrand. Since $h(n)1\otimes\delta_\mu = 0$ for $n>0$ so that Ω_μ is fixed by $E^+(\alpha,z)$, we have

$$X_\alpha(n)1\otimes\delta_\mu = \frac{\epsilon(\alpha,\mu)}{2\pi i}\oint\frac{dz}{z}z^{n+\langle\alpha,\mu\rangle+\frac{\langle\alpha,\alpha\rangle}{2}}E^-(\alpha,z)1\otimes\delta_{\alpha+\mu} \tag{4.25}$$

The coefficient of z^k in the power series expansion of $E^-(\alpha,z)=\exp\Big(\sum_{m>0}\frac{\alpha(-m)}{m}z^m\Big)$ is 0 if $k<0$ and otherwise of the form $\frac{\alpha(-1)^k}{k!}$ plus terms involving powers of $\alpha(-1)$ strictly smaller than k so that it does not vanish if $k\geq 0$. Since the map $p\rightarrow p\cdot 1$, where p is any polynomial in the $h(n)$ with $n>0$, is an isomorphism, we conclude that (4.25) vanishes iff

$$n+\langle\alpha,\mu\rangle+\frac{\langle\alpha,\alpha\rangle}{2}\geq 1 \tag{4.26}$$

If $n>0$, this is equivalent to $\langle\alpha,\mu\rangle\geq -1$ for any α *i.e.* (replacing α by $-\alpha$) to $|\langle\alpha,\mu\rangle|\leq 1$ so that μ is a minimal weight. Choosing now $n=0$ and $\alpha>0$ yields $\langle\alpha,\mu\rangle\geq 0$ so that μ is dominant and is therefore the unique minimal dominant weight in $\nu+\Lambda_R$. It follows that $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ is the direct sum of all level 1 representations of $L^{\rm pol}\mathfrak{g}$ since these are in bijective correspondence with Λ_R cosets in Λ_W $\diamond$

Remark. As remarked in the introduction, the vertex operator construction does not provide one with a natural action of LG on the Hilbert space completion of $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$. The infinitesimal action of $L^{\rm pol}\mathfrak{g}$ can however be exponentiated to LG by using analytic methods [**GoWa, TL1**]. When $G=\mathrm{Spin}_{2n}$ (or SU_n), one may alternatively resort to the Fermionic realisation of the level 1 representations described in chapter III. The latter exponentiates to LG by construction and the isomorphism of $L^{\rm pol}\mathfrak{g}$–modules may be used to transport the action of LG on the fermionic Fock spaces to one on the completion of $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$.

5. The level 1 primary fields

Having realised the finite energy subspaces of the level 1 representations of LG as summands of $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$, we construct in this section the level 1 primary fields by following a scheme similar to the one adopted for the vertex operators $X_\alpha(z)$. We obtain in §5.2 their equivariance properties with respect to LT. The calculation relies upon the assumption that the primary fields extend to continuous operator–valued distributions and therefore has heuristic value only. We use it nonetheless to deduce their explicit form and then show in §5.3 that the guesses have the correct commutation relations with $L^{\rm pol}\mathfrak{g}$.

The form of the primary fields resembles that of the $X_\alpha(z)$ with an additional term accounting for the non–integrality of the weight lattice. In §5.5, we give the complete list of simply–laced G for which this correction factor cannot altogether be dispensed with. It comprises SU_2 and Spin_{2n} with n not divisible by 4.

Finally, in §5.4, we study the level 1 primary fields for $L\,\mathrm{Spin}_{2n}$. We consider the vector primary field and recover the result of chapter III that it is a Fermionic field. We also prove that all level 1 primary fields for $L\,\mathrm{Spin}_8$ are Fermi fields.

5.1. Action of $\check{T}$ and L_0 on $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$.

We begin by identifying explicitly the operators giving the action of $\check{T}\subset LT$ and L_0 on $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$. Notice that the latter supports two projective representations of $\check{T}$. The first is given by (4.4) and extends to one of Λ_W while the second is obtained by restricting the representation π of LG to $\check{T}\subset LT$. Surprisingly perhaps, these do not coincide. In fact the following is true

Proposition 5.1.1. *For any $\alpha\in\Lambda_R$, the following holds in $PU(\mathcal{S}\otimes\mathbb{C}[\Lambda_W])$*

$$\pi(\zeta_\alpha)=U_\alpha\chi_\alpha \tag{5.1.1}$$

where $\chi_\alpha\in\mathrm{End}(\mathbb{C}[\Lambda_W])$ acts as multiplication by the function $\chi_\alpha(\mu)=\dfrac{(-1)^{\langle\alpha,\mu\rangle}}{\widetilde{\omega}(\alpha,\mu)}$.

Proof. By construction, the operators U_μ, $\mu\in\Lambda_W$ satisfy the hypothesis of proposition 2.3.2. Indeed, $U_\mu h(n)U_\mu^*=h(n)-\delta_{n0}\langle\mu,h\rangle$ and, by (4.8), $U_\mu X_\alpha(z)U_\mu^*=z^{-\langle\alpha,\mu\rangle}X_\alpha(z)$. They therefore satisfy

$$U_\mu\pi(\zeta_\alpha)U_\mu^*\pi(\zeta_\alpha)^*=(-1)^{\langle\mu,\alpha\rangle} \tag{5.1.2}$$

On the other hand, by (4.5),

$$U_\mu U_\alpha U_\mu U_\alpha^* = \widetilde{\omega}(\mu,\alpha) \tag{5.1.3}$$

and therefore (5.1.2) continues to hold if $\pi(\zeta_\alpha)$ is replaced by $W_\alpha = U_\alpha X_\alpha$. It follows that the operators $C_\alpha = W_\alpha \pi(\zeta_\alpha)^*$ commute with the U_μ. By proposition 2.2.2, they also commute with the action of $L^{\mathrm{pol}}\mathfrak{t}$ and therefore act as scalars $\diamond$

The following useful result may be found in [**GO2**, §4.3]

LEMMA 5.1.2. *Let V be an irreducible G–module with Casimir C_V. If $\Pi(V)$ is the set of weights of V counted with multiplicities, then*

$$\frac{\dim V}{\sum_{\mu\in\Pi(V)}\|\mu\|^2} C_V = \frac{\dim G}{\operatorname{rank} G} \tag{5.1.4}$$

In particular, if G is simply–laced and V is a minimal representation with highest weight λ, the level 1 conformal dimension of V is equal to

$$\Delta_V := \frac{C_V}{2+C_{\mathfrak{g}}} = \frac{1}{2}\|\lambda\|^2 \tag{5.1.5}$$

PROOF. The map $X\otimes Y \to \operatorname{tr}_V(XY)$ is a symmetric, bilinear and ad–invariant form on $\mathfrak{g}_{\mathbb{C}}$ so that

$$\operatorname{tr}_V(XY) = \alpha_V \langle X, Y\rangle \tag{5.1.6}$$

where $\langle\cdot,\cdot\rangle$ is the basic inner product and $\alpha_V\in\mathbb{C}$. The proportionality constant α_V may be evaluated in two different ways. Choosing basis X_k, X^k of $\mathfrak{g}_{\mathbb{C}}$ dual for $\langle\cdot,\cdot\rangle$, we get

$$\dim(V) C_V = \operatorname{tr}(X_k X^k) = \alpha_V \dim(G) \tag{5.1.7}$$

On the other hand, if h_i, h^i are dual basis of $\mathfrak{t}_{\mathbb{C}}$, we find by evaluating the trace in a basis of weight vectors that

$$\sum_{\mu\in\Pi(V)} \|\mu\|^2 = \operatorname{tr}(h_i h^i) = \alpha_V \operatorname{rank}(G) \tag{5.1.8}$$

Eliminating α_V yields (5.1.4). If V is minimal, all its weights lie on the orbit of λ under the Weyl group and have multiplicity one by proposition I.2.2.1. Thus, $\sum_{\mu\in\Pi(V)}\|\mu\|^2 = \dim(V)\|\lambda\|^2$. Moreover, if G is simply–laced, we find from (5.1.4)

$$C_{\mathfrak{g}} = 2\frac{\dim(G)-\operatorname{rank}(G)}{\operatorname{rank}(G)} \tag{5.1.9}$$

and therefore (5.1.5) $\diamond$

Let now L_0 be the infinitesimal generator of rotations given by the Segal–Sugawara formula

$$L_0 = \frac{1}{\kappa}\Big(\frac{1}{2}X_i(0)X^i(0) + \sum_{m>0} X_i(-m)X^i(m)\Big) \tag{5.1.10}$$

where $\kappa = 1+\frac{C_{\mathfrak{g}}}{2}$. Then

COROLLARY 5.1.3. *The action of the infinitesimal generator of rotations L_0 on $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ is given by*

$$L_0 = \frac{1}{2}h_k h^k + \sum_{m>0} h_k(-m)h^k(m) \tag{5.1.11}$$

where h_k, h^k are dual basis of $\mathfrak{t}_{\mathbb{C}}$ for the basic inner product. In particular,

$$L_0\, 1\otimes\delta_\mu = \frac{1}{2}\langle\mu,\mu\rangle 1\otimes\delta_\mu \tag{5.1.12}$$

PROOF. Let D be operator defined by the right–hand side of (5.1.11). It satisfies by construction $[D, X(n)] = -nX(n)$ and therefore differs from L_0 by an additive constant on each irreducible summand $\mathcal{S}\otimes\mathbb{C}[\mu+\Lambda_R]$. Choosing μ dominant minimal, L_0 acts on the corresponding highest weight vector $1\otimes\delta_\mu$ as multiplication by the conformal dimension of the lowest energy subspace of $\mathbb{C}[\mu+\Lambda_R]$ and therefore, by the previous lemma, by $\frac{\langle\mu,\mu\rangle}{2}$. Since $D1\otimes\delta_\mu = \frac{\langle\mu,\mu\rangle}{2}$, the two operators coincide $\diamond$

5.2. LT–equivariance of primary fields.

Assuming that the level 1 primary fields extend to operator–valued distributions, we derive below their commutation relations with LT. Although the continuity properties of these fields will only be established in chapter VI, the present discussion serves as motivation for the next subsection.

By proposition I.4.1, the charges of the level 1 primary fields are necessarily admissible at level 1 and are therefore the minimal G–modules since G is simply–laced. Fix one such V_k and let $\phi_{ji} : \mathcal{H}_i^{\text{fin}} \otimes V_k[z, z^{-1}] \to \mathcal{H}_j^{\text{fin}}$ be a primary field of charge V_k. Since $\mathcal{H}_i^{\text{fin}}, \mathcal{H}_j^{\text{fin}}$ are summands of $\mathcal{H}^{\text{fin}} = \mathcal{S} \otimes \mathbb{C}[\Lambda_W]$, we regard ϕ_{ji} as a map $\mathcal{H}^{\text{fin}} \otimes V_k[z, z^{-1}] \to \mathcal{H}^{\text{fin}}$ by extending it by zero. By definition, it satisfies

$$[\pi(X), \phi_{ji}(f)] = \phi_{ji}(Xf) \tag{5.2.1}$$

for any $X \in L^{\text{pol}}\mathfrak{g}$ and $f \in V_k[z, z^{-1}]$. We assume in this subsection that ϕ_{ji} extends to a jointly continuous map $\mathcal{H}^\infty \otimes C^\infty(S^1, V_k) \to \mathcal{H}^\infty$. It follows by continuity that (5.2.1) holds on $\mathcal{H}^\infty$ for any $X \in L\mathfrak{g}$ and $f \in C^\infty(S^1, V_k)$ and therefore that, for any $\gamma \in LG$

$$\pi(\gamma)\phi_{ji}(f)\pi(\gamma)^* = \phi_{ji}(\gamma f) \tag{5.2.2}$$

To see this, it is sufficient to consider the case $\gamma = \exp_{LG}(X)$ since LG is generated by the image of the exponential map. Let $F(t) = e^{-t\pi(X)}\phi_{ji}(e^{tX}f)e^{t\pi(X)}\xi$ where $\xi \in \mathcal{H}^\infty$. The LG–invariance of $\mathcal{H}^\infty$ (proposition II.1.5.3) and (5.2.1) imply that F is differentiable and that $\dot{F} = 0$ whence $F(1) \equiv F(0) = \phi_{ji}(f)\xi$.

We now restrict our attention to $\gamma \in LT$ and rephrase (5.2.2) in terms of the generating function $\phi_{ji}(z) = \sum_{n\in\mathbb{Z}} \phi_{ji}(v, n) z^{-n-(\Delta_i+\Delta_k-\Delta_j)}$ where, as customary $\phi_{ji}(v, n) = \phi_{ji}(v \otimes e^{in\theta})$ and the $\Delta_\cdot$ are the conformal dimensions of $\mathcal{H}_i(0), V_k$ and $\mathcal{H}_j(0)$ respectively. Let $v_\mu \in V_k$ be a vector of weight $\mu \in \Lambda_W$ and, for $n \in \mathbb{N}$ set $v_\mu(n) = e^{in\theta} \otimes v_\mu \in C^\infty(S^1, V_k)$. Clearly, for $h \in \mathfrak{t}_\mathbb{C}$ and $\tau \in T$,

$$h(m)v_\mu(n) = \langle h, \mu\rangle v_\mu(n+m) \qquad \text{and} \qquad \tau v_\mu(n) = \mu(\tau)v_\mu(n) \tag{5.2.3}$$

Moreover, since $\zeta_\alpha(\theta) = \exp(-i\alpha\theta)$, we find $\zeta_\alpha v_\mu(n) = v_\mu(n - \langle\alpha, \mu\rangle)$. Thus, in terms of the formal power series $\phi_{ji}(v_\mu, z) = \sum_{n\in\mathbb{Z}} \phi(v_\mu(n))z^{-n}$, we have

$$[h(n), \phi_{ji}(v_\mu, z)] = \langle h, \mu\rangle z^n \phi_{ji}(v_\mu, z) \tag{5.2.4}$$

$$\pi(\tau)\phi_{ji}(v_\mu, z)\pi(\tau)^* = \mu(\tau)\phi_{ji}(v_\mu, z) \tag{5.2.5}$$

$$\pi(\zeta_\alpha)\phi_{ji}(v_\mu, z)\pi(\zeta_\alpha)^* = z^{-\langle\alpha,\mu\rangle}\phi_{ji}(v_\mu, z) \tag{5.2.6}$$

and, by (5.1.5)

$$[L_0, \phi_{ji}(v_\mu, z)] = (z\frac{d}{dz} + \Delta_k)\phi_{ji}(v_\mu, z) = (z\frac{d}{dz} + \frac{\langle\mu, \mu\rangle}{2})\phi_{ji}(v_\mu, z) \tag{5.2.7}$$

5.3. The construction of level 1 primary fields.

We turn now to the construction of the primary fields $\phi_{ji}(z)$. It will be convenient to consider all fields with a given charge V_k at once by working with $\Phi(z) = \bigoplus_{j,i} \phi_{ji}(z)$. Since V_k is minimal, the spaces $\text{Hom}(\mathcal{H}_i(0) \otimes V_k, \mathcal{H}_j(0))$ are at most one–dimensional by corollary I.2.2.3 and the individual $\phi_{ji}(z)$ may be recovered by sandwiching $\Phi(z)$ between the orthogonal projections P_j, P_i onto $\mathcal{H}_j^{\text{fin}}, \mathcal{H}_i^{\text{fin}} \subset \mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ respectively. Clearly, if $v_\mu \in V_k$ is a weight vector of weight μ, $\Phi_\mu(z) = \Phi(v_\mu, z)$ satisfies (5.2.4)–(5.2.7).

To determine the form of $\Phi_\mu(z)$ from these commutation relations, fix for any coset $\mu + \Lambda_R \subset \Lambda_W$ a representative $[\mu] \in \Lambda_W$ and set $[\alpha] = 0$ for any $\alpha \in \Lambda_R$. Let $\mu \in \Lambda_W$ and set

$$\Phi_\mu(z) = E^-(\mu, z)V_\mu z^{\mu+\frac{\langle\mu,\mu\rangle}{2}} \frac{\widetilde{\omega}(\mu, \cdot - [\cdot])}{e^{i\pi\langle\mu, \cdot-[\cdot]\rangle}} E^+(\mu, z) \tag{5.3.1}$$

It is easy to check, using proposition (5.1.1) that Φ_μ satisfies (5.2.4)–(5.2.6). Moreover, by corollary 5.1.3,

$$[L_0, \Phi_\mu(z)] = z\frac{d}{dz}\Phi_\mu(z) \tag{5.3.2}$$

and we should therefore be considering $\Phi_\mu(z)z^{-\frac{\langle\mu,\mu\rangle}{2}}$ instead. By analogy with the expressions for the vertex operators $X_\alpha(z)$, we shall retain the erroneous $z^{\frac{\langle\mu,\mu\rangle}{2}}$ instead. We show below that, as μ spans the weights of V_k, the $\Phi_\mu(z)$ describe the components of a primary field of charge V_k. We shall need for this purpose an explicit description of the infinitesimal action of $\mathfrak{g}$ on any minimal G–module.

LEMMA 5.3.1. *Let $\{\mu\} \subset \Lambda_W$ be the collection of weights of minimal length in a given Λ_R–coset. Then, the subspace of $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ spanned by the vectors $1\otimes\delta_\mu$ is invariant and irreducible under the action of $\mathfrak{g}$ given by the operators $X_\alpha(0)$ and h. It is therefore isomorphic to the minimal G–module V_λ whose highest weight λ is the unique dominant element in $\{\mu\}$. The $\mathfrak{g}$–action is explicitly given by*

$$h\,1\otimes\delta_\mu = \langle h,\mu\rangle 1\otimes\delta_\mu \tag{5.3.3}$$

$$X_\alpha(0)\,1\otimes\delta_\mu = \begin{cases} \epsilon(\mu,\alpha)1\otimes\delta_{\alpha+\mu} & \textit{if } \langle\alpha,\mu\rangle = -1 \quad \textit{i.e. iff } \|\alpha+\mu\| = \|\mu\| \\ 0 & \textit{if } \langle\alpha,\mu\rangle \geq 0 \quad \textit{i.e. iff } \|\alpha+\mu\| > \|\mu\| \end{cases} \tag{5.3.4}$$

PROOF. Let μ be of minimal length so that $\|\mu\pm\alpha\|^2 \geq \|\mu\|^2$ for any root α and therefore $\langle\mu,\alpha\rangle \in \{-1,0,1\}$. Proceeding as in the proof of theorem 4.3, we find

$$\begin{aligned} X_\alpha(0)\,1\otimes\delta_\mu &= \frac{1}{2\pi i}\oint\frac{dz}{z}X_\alpha(z)1\otimes\delta_\mu \\ &= \frac{\epsilon(\mu,\alpha)}{2\pi i}\oint\frac{dz}{z}z^{\langle\alpha,\mu\rangle+\frac{\langle\alpha,\alpha\rangle}{2}}E^-(\alpha,z)1\otimes\delta_{\alpha+\mu} \qquad (5.3.5) \\ &= \begin{cases} \epsilon(\mu,\alpha)1\otimes\delta_{\alpha+\mu} & \text{if } \langle\alpha,\mu\rangle = -1 \quad \textit{i.e.} \text{ iff } \|\alpha+\mu\| = \|\mu\| \\ 0 & \text{if } \langle\alpha,\mu\rangle \geq 0 \quad \textit{i.e.} \text{ iff } \|\alpha+\mu\| > \|\mu\| \end{cases} \end{aligned}$$

Thus, the span of $1\otimes\delta_\mu$ with μ varying in the vectors of minimal length in a given Λ_R–coset is invariant under the $X_\alpha(0)$. Since its weights coincide by proposition I.2.2.1 with those of V_λ the two representations have the same character and are therefore isomorphic $\diamond$

THEOREM 5.3.2. *Let V be a minimal G–module. Then V has a basis of weight vectors v_μ such that the assignement*

$$v_\mu \longrightarrow \Phi_\mu(z) = E^-(\mu,z)V_\mu z^{\mu+\frac{\langle\mu,\mu\rangle}{2}}\frac{\widetilde{\omega}(\mu,\cdot-[\cdot])}{e^{i\pi\langle\mu,\cdot-[\cdot]\rangle}}E^+(\mu,z) \tag{5.3.6}$$

is the direct sum of all level 1 primary fields of charge V.

PROOF. Set $\eta_\mu(\nu) = \dfrac{\widetilde{\omega}(\mu,\nu-[\nu])}{e^{i\pi\langle\mu,\nu-[\nu]\rangle}}$ and notice that $\eta_{\mu+\alpha} = \eta_\mu$ for any $\alpha\in\Lambda_R$ since $\widetilde{\omega}(\alpha,\beta) = (-1)^{\langle\alpha,\beta\rangle}$ for any $\alpha,\beta\in\Lambda_R$. Moreover, $[\nu+\alpha] = [\nu]$ and $[\alpha] = 0$ for $\alpha\in\Lambda_R$ implies that $\eta_\mu(\nu+\alpha) = \eta_\mu(\nu)\eta_\mu(\alpha)$. We now have

$$X_\alpha(z)\Phi_\mu(\zeta) = \epsilon(\mu,\alpha)\Big(1-\frac{\zeta}{z}\Big)^{\langle\alpha,\mu\rangle}\Big(\frac{z}{\zeta}\Big)^{\frac{\langle\alpha,\mu\rangle}{2}}V_{\alpha+\mu}z^{\alpha+\frac{\langle\alpha,\alpha\rangle}{2}+\frac{\langle\alpha,\mu\rangle}{2}}\zeta^{\mu+\frac{\langle\mu,\mu\rangle}{2}+\frac{\langle\alpha,\mu\rangle}{2}}\eta_\mu\mathcal{R}(\alpha,\mu,z,\zeta) \tag{5.3.7}$$

where

$$\mathcal{R}(\alpha,\mu,z,\zeta) = E^-(\alpha,z)E^-(\mu,\zeta)E^+(\alpha,z)E^+(\mu,\zeta) = \mathcal{R}(\mu,\alpha,\zeta,z) \tag{5.3.8}$$

On the other hand,

$$\Phi_\mu(\zeta)X_\alpha(z) = \epsilon(\alpha,\mu)\eta_\mu(\alpha)\Big(1-\frac{z}{\zeta}\Big)^{\langle\alpha,\mu\rangle}\Big(\frac{\zeta}{z}\Big)^{\frac{\langle\alpha,\mu\rangle}{2}}V_{\mu+\alpha}z^{\alpha+\frac{\langle\alpha,\alpha\rangle}{2}+\frac{\langle\alpha,\mu\rangle}{2}}\zeta^{\mu+\frac{\langle\mu,\mu\rangle}{2}+\frac{\langle\alpha,\mu\rangle}{2}}\eta_\mu\mathcal{R}(\alpha,\mu,z,\zeta) \tag{5.3.9}$$

Thus, since

$$\epsilon(\alpha,\mu)\eta_\mu(\alpha) = \epsilon(\mu,\alpha)\widetilde{\omega}(\alpha,\mu)\frac{\widetilde{\omega}(\mu,\alpha)}{(-1)^{\mu,\alpha}} = \epsilon(\alpha,\mu)(-1)^{\langle\alpha,\mu\rangle} \tag{5.3.10}$$

we get

$$[X_\alpha(z),\Phi_\mu(\zeta)]=\epsilon(\mu,\alpha)\mathcal{D}(z,\zeta)V_{\mu+\alpha}z^{\alpha+\frac{\langle\alpha,\alpha\rangle}{2}+\frac{\langle\alpha,\mu\rangle}{2}}\zeta^{\mu+\frac{\langle\mu,\mu\rangle}{2}+\frac{\langle\alpha,\mu\rangle}{2}}\eta_\mu\mathcal{R}(\alpha,\mu,z,\zeta) \tag{5.3.11}$$

where

$$\begin{aligned}\mathcal{D}(z,\zeta)&=\Big(\frac{\zeta}{z}\Big)^{-\frac{\langle\alpha,\mu\rangle}{2}}\left[\Big(1-\frac{\zeta}{z}\Big)^{\langle\alpha,\mu\rangle}-(-1)^{\langle\alpha,\mu\rangle}\Big(1-\frac{z}{\zeta}\Big)^{\langle\alpha,\mu\rangle}\Big(\frac{\zeta}{z}\Big)^{\langle\alpha,\mu\rangle}\right]\\&=\begin{cases}0&\text{if }\langle\alpha,\mu\rangle\geq 0\\ \Big(\frac{\zeta}{z}\Big)^{\frac{1}{2}}\delta\Big(\frac{\zeta}{z}\Big)&\text{if }\langle\alpha,\mu\rangle=-1\end{cases}\end{aligned} \tag{5.3.12}$$

In the latter case, we get by (1.10), the fact that $\mathcal{R}$ contains only integral powers of z,ζ and that α has integral eigenvalues

$$\begin{aligned}[X_\alpha(z),\Phi_\mu(\zeta)]&=\epsilon(\mu,\alpha)\Big(\frac{\zeta}{z}\Big)^{\frac{1}{2}}\delta\Big(\frac{\zeta}{z}\Big)V_{\mu+\alpha}z^{\alpha+\frac{\langle\alpha,\alpha\rangle}{2}+\frac{\langle\alpha,\mu\rangle}{2}}\zeta^{\mu+\frac{\langle\mu,\mu\rangle}{2}+\frac{\langle\alpha,\mu\rangle}{2}}\eta_\mu\mathcal{R}(\alpha,\mu,z,\zeta)\\&=\epsilon(\mu,\alpha)\delta\Big(\frac{\zeta}{z}\Big)V_{\mu+\alpha}\zeta^{\alpha+\mu+\frac{\langle\alpha+\mu,\alpha+\mu\rangle}{2}}\eta_{\mu+\alpha}\mathcal{R}(\alpha,\mu,\zeta,\zeta)\\&=\epsilon(\mu,\alpha)\delta\Big(\frac{\zeta}{z}\Big)\Phi_{\alpha+\mu}(\zeta)\end{aligned} \tag{5.3.13}$$

The theorem now follows from lemma 5.3.1 $\diamond$

Remark. The additional factor $\widetilde{\omega}(\mu,\cdot-[\cdot])e^{-i\pi\langle\mu,\cdot-[\cdot]\rangle}$ used in the definition of the primary field is required to ensure that $\Phi_\mu(z)$ has the correct commutation relations with the $X_\alpha(z)$. It can only be dispensed with if Λ_W, or less ambitiously the intermediate lattice $\Lambda_R\subset\Lambda\subset\Lambda_W$ containing the collection of weights $\{\mu\}$ of a given primary field possesses a skew–symmetric form $\widetilde{\omega}$ satisfying $\omega(\mu,\alpha)=(-1)^{\langle\mu,\alpha\rangle}$ whenever $\mu\in\Lambda$ and $\alpha\in\Lambda_R$. Such a form need not exist. For example, when $G=\mathrm{SU}_2$, the weight lattice is cyclic and therefore does not possess non–trivial skew–symmetric forms. On the other hand, a suitable $\widetilde{\omega}$ would satisfy $\widetilde{\omega}(\alpha,\frac{\alpha}{2})=(-1)^{\langle\alpha,\frac{\alpha}{2}\rangle}=-1$ where α is the positive root of SU_2 and $\frac{\alpha}{2}$ the corresponding generator of the weight lattice. In §5.5, we give the complete list of intermediate lattices Λ possessing such an ω. We simply note here that it comprises integral lattices, since in that case $\widetilde{\omega}(\lambda,\mu)=(-1)^{\langle\lambda,\mu\rangle+\langle\lambda,\lambda\rangle\langle\mu,\mu\rangle}$ has the required properties. This fact will be exploited below to study the vector primary field for $L\,\mathrm{Spin}_{2n}$.

5.4. The level 1 primary fields of $L\,\mathrm{Spin}_{2n}$.

We consider now the case $G=\mathrm{Spin}_{2n}$ and rederive the result obtained in chapter III that the level 1 vector primary field for $L\,\mathrm{Spin}_{2n}$ is a Fermi field. Its construction within the vertex operator model is an instance of the celebrated Boson–Fermion correspondence.

Lemma 5.4.1. *Let $\Lambda_R\subset\Lambda\subset\Lambda_W$ be an intermediate lattice which is integral and choose $\widetilde{\omega}$ such that*

$$\widetilde{\omega}(\lambda,\mu)=(-1)^{\langle\lambda,\mu\rangle+\langle\lambda,\lambda\rangle\langle\mu,\mu\rangle} \tag{5.4.1}$$

Then, if $\lambda,\mu\in\Lambda$ are of norm one

$$\{\Phi_\lambda(z),\Phi_\mu(\zeta)\}=\begin{cases}\big(\frac{\zeta}{z}\big)^{-\lambda+\frac{\langle\lambda,\lambda\rangle}{2}}\delta\big(\frac{\zeta}{z}\big)&\textit{if }\lambda+\mu=0\\ 0&\textit{otherwise}\end{cases} \tag{5.4.2}$$

Proof. Notice that the skew–symmetric form given by (5.4.1) is well–defined and bilinear since Λ is integral and therefore $\|\sum\lambda_i\|^2=\sum\|\lambda_i\|^2 \mod 2$. Moreover, since Λ_R is even, $\omega(\mu,\alpha)=(-1)^{\langle\mu,\alpha\rangle}$ whenever $\mu\in\Lambda$ and $\alpha\in\Lambda_R$ and it follows that (5.3.1) reduces to

$$\Phi_\mu(z)=E^-(\mu,z)V_\mu z^{\mu+\frac{\langle\mu,\mu\rangle}{2}}E^+(\mu,z) \tag{5.4.3}$$

Thus,

$$\Phi_\lambda(z)\Phi_\mu(\zeta) = \epsilon(\mu,\lambda)\Big(1-\frac{\zeta}{z}\Big)^{\langle\lambda,\mu\rangle}\Big(\frac{z}{\zeta}\Big)^{\frac{\langle\lambda,\mu\rangle}{2}} V_{\lambda+\mu} z^{\lambda+\frac{\langle\lambda,\lambda\rangle}{2}+\frac{\langle\lambda,\mu\rangle}{2}}\zeta^{\mu+\frac{\langle\mu,\mu\rangle}{2}+\frac{\langle\lambda,\mu\rangle}{2}}\mathcal{R}(\lambda,\mu,z,\zeta) \tag{5.4.4}$$

where

$$\mathcal{R}(\lambda,\mu,z,\zeta) = E^-(\lambda,z)E^-(\mu,\zeta)E^+(\lambda,z)E^+(\mu,\zeta) = \mathcal{R}(\mu,\lambda,\zeta,z) \tag{5.4.5}$$

so that

$$\Phi_\mu(\zeta)\Phi_\lambda(z) = \epsilon(\lambda,\mu)\Big(1-\frac{z}{\zeta}\Big)^{\langle\lambda,\mu\rangle}\Big(\frac{\zeta}{z}\Big)^{\frac{\langle\lambda,\mu\rangle}{2}} V_{\lambda+\mu} z^{\lambda+\frac{\langle\lambda,\lambda\rangle}{2}+\frac{\langle\lambda,\mu\rangle}{2}}\zeta^{\mu+\frac{\langle\mu,\mu\rangle}{2}+\frac{\langle\lambda,\mu\rangle}{2}}\mathcal{R}(\lambda,\mu,z,\zeta) \tag{5.4.6}$$

Since $\epsilon(\lambda,\mu) = \epsilon(\mu,\lambda)(-1)^{\langle\lambda,\mu\rangle+\langle\lambda,\lambda\rangle\langle\mu,\mu\rangle} = -(-1)^{\langle\lambda,\mu\rangle}\epsilon(\mu,\lambda)$, we get

$$\{\Phi_\lambda(z),\Phi_\mu(\zeta)\} = \epsilon(\mu,\lambda)\mathcal{D}(z,\zeta)V_{\lambda+\mu}z^{\lambda+\frac{\langle\lambda,\lambda\rangle}{2}+\frac{\langle\lambda,\mu\rangle}{2}}\zeta^{\mu+\frac{\langle\mu,\mu\rangle}{2}+\frac{\langle\lambda,\mu\rangle}{2}}\mathcal{R}(\lambda,\mu,z,\zeta) \tag{5.4.7}$$

where, using (1.8)

$$\begin{aligned}\mathcal{D}(z,\zeta) &= \Big(\frac{\zeta}{z}\Big)^{-\frac{\langle\lambda,\mu\rangle}{2}}\left[\Big(1-\frac{\zeta}{z}\Big)^{\langle\lambda,\mu\rangle} - (-1)^{\langle\lambda,\mu\rangle}\Big(1-\frac{z}{\zeta}\Big)^{\langle\lambda,\mu\rangle}\Big(\frac{z}{\zeta}\Big)^{-\langle\lambda,\mu\rangle}\right]\\ &= \begin{cases} 0 & \text{if } \langle\lambda,\mu\rangle \geq 0\\ \Big(\frac{\zeta}{z}\Big)^{\frac{1}{2}}\delta\Big(\frac{\zeta}{z}\Big) & \text{if } \langle\lambda,\mu\rangle = -1\end{cases}\end{aligned} \tag{5.4.8}$$

Since I is an integral lattice and λ,μ are of norm one, $\langle\lambda,\mu\rangle\in\{-1,0,1\}$ with $\langle\lambda,\mu\rangle=-1$ iff $\mu=-\lambda$. In the latter case, we get using (1.10) and the fact that $\mathcal{R}$ only contains integral powers of z and ζ,

$$\delta\Big(\frac{\zeta}{z}\Big)\mathcal{R}(\lambda,\mu,z,\zeta) = \delta\Big(\frac{\zeta}{z}\Big)\mathcal{R}(\lambda,-\lambda,\zeta,\zeta) = 1 \tag{5.4.9}$$

Moreover, since ϵ is normalised, $\epsilon(\lambda,-\lambda)=1$ and $V_0=1$. Thus, if $\lambda=-\mu$,

$$\{\Phi_\lambda(z),\Phi_{-\lambda}(\zeta)\} = \Big(\frac{\zeta}{z}\Big)^{\frac{1}{2}}\delta\Big(\frac{\zeta}{z}\Big)z^\lambda\zeta^{-\lambda} \tag{5.4.10}$$

as claimed $\diamond$

The following is well–known [**GO2**, §7.1]

PROPOSITION 5.4.2.

(i) *The level 1 primary fields of* $L\,\mathrm{Spin}_{2n}$ *whose charge is the vector representation are Fermi fields.*

(ii) *All level 1 primary fields of* $L\,\mathrm{Spin}_8$ *are Fermi fields.*

PROOF. (i) The root lattice of Spin_{2n} is the $\mathbb{Z}$–span in $\mathbb{R}^n$ of the vectors $\theta_i-\theta_{i+1}$, $i=1\dots n-1$ and $\theta_{n-1}+\theta_n$ and is therefore the lattice of integral points with even sum of coordinates. The dual lattice Λ_W is $I+\frac{1}{2}(\theta_1+\cdots+\theta_n)$ where $I=\mathbb{Z}^n$ is the integral lattice. The minimal dominant weights are $v=\theta_1$ and $s_\pm=\frac{1}{2}(\theta_1+\cdots+\theta_{n-1}\pm\theta_n)$ and correspond to the vector and spin modules respectively. Let now $\lambda\in v+\Lambda_R=I$ of norm one so that $\lambda=\pm\theta_i$ and the corresponding operator on $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ has integral eigenvalues on

$$\mathcal{F}_{NS}^{\mathrm{fin}}=\mathcal{S}\otimes\mathbb{C}[I]=\mathcal{S}\otimes\mathbb{C}[\Lambda_R]\bigoplus\mathcal{S}\otimes\mathbb{C}[v+\Lambda_R] \tag{5.4.11}$$

and half integral eigenvalues (*i.e.* elements of $\frac{1}{2}+\mathbb{Z}$) on

$$\mathcal{F}_{R}^{\mathrm{fin}}=\mathcal{S}\otimes\mathbb{C}[s_++I]=\mathcal{S}\otimes\mathbb{C}[s_++\Lambda_R]\bigoplus\mathcal{S}\otimes\mathbb{C}[s_-+\Lambda_R] \tag{5.4.12}$$

The notation $\mathcal{F}_{NS}^{\mathrm{fin}}$ and $\mathcal{F}_{R}^{\mathrm{fin}}$ refers to the fact that, by theorem 4.3 and and proposition III.3.3.1 these subspaces are isomorphic, as $L^{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}$–modules to the finite energy subspaces of the Neveu–Schwarz and Ramond Fermionic Fock spaces constructed in chapter III. By (1.10)

$$\left(\frac{\zeta}{z}\right)^{-\lambda+\frac{\langle\lambda,\lambda\rangle}{2}}\delta\left(\frac{\zeta}{z}\right)=\begin{cases}\left(\frac{\zeta}{z}\right)^{\frac{1}{2}}\delta\left(\frac{\zeta}{z}\right) & \text{on } \mathcal{F}_{NS}^{\mathrm{fin}}\\ \delta\left(\frac{\zeta}{z}\right) & \text{on } \mathcal{F}_{R}^{\mathrm{fin}}\end{cases} \tag{5.4.13}$$

Moreover, the modes of Φ_λ clearly preserve the splitting $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]=\mathcal{F}_{NS}^{\mathrm{fin}}\bigoplus\mathcal{F}_{R}^{\mathrm{fin}}$. Thus, defining, for any i, $\Phi_{\theta_i}(z)=\psi_i^{\mathrm{NS}}(z)+\psi_i^{\mathrm{R}}(z)$ where

$$\psi_i^{\mathrm{NS}}(z)=\sum_{n\in\frac{1}{2}+\mathbb{Z}}\Phi_{\theta_i}(n)z^{-n} \qquad \psi_i^{\mathrm{R}}(z)=\sum_{n\in\mathbb{Z}}\Phi_{\theta_i}(n)z^{-n} \tag{5.4.14}$$

and noticing that the formal adjunction property $\Phi_{\theta_i}(z)^*=\Phi_{-\theta_i}(z)$ is inherited by $\psi_i^{\mathrm{NS}}(z)$ and $\psi_i^{\mathrm{R}}(z)$ so that $\psi_{-i}^{\mathrm{NS}}(-n)=\psi_i^{\mathrm{NS}}(n)^*$, lemma 5.4.1 and (5.4.13) imply that

$$\{\psi_i^{\mathrm{NS}}(n),\psi_j^{\mathrm{NS}}(m)\}=0=\{\psi_i^{\mathrm{R}}(n),\psi_j^{\mathrm{R}}(m)\} \tag{5.4.15}$$

and

$$\{\psi_i^{\mathrm{NS}}(n),\psi_j^{\mathrm{NS}}(m)^*\}=\begin{cases}\delta_{nm}\delta_{ij} & \text{on } \mathcal{F}_{NS}^{\mathrm{fin}}\\ 0 & \text{on } \mathcal{F}_{R}^{\mathrm{fin}}\end{cases} \qquad \{\psi_i^{\mathrm{R}}(n),\psi_j^{\mathrm{R}}(m)^*\}=\begin{cases}0 & \text{on } \mathcal{F}_{NS}^{\mathrm{fin}}\\ \delta_{nm}\delta_{ij} & \text{on } \mathcal{F}_{R}^{\mathrm{fin}}\end{cases} \tag{5.4.16}$$

and in particular $\psi_i^{\mathrm{NS}}(z)=0$ on $\mathcal{F}_{R}^{\mathrm{fin}}$ and $\psi_i^{\mathrm{R}}(z)=0$ on $\mathcal{F}_{NS}^{\mathrm{fin}}$
(ii) A similar analysis holds for all primary fields of $L\,\mathrm{Spin}_8$ since in that each of the cosets $v+\Lambda_R$, $s_\pm+\Lambda_R$ are integral $\diamond$

REMARK. The same computation as that performed in the proof of theorem 4.2 shows that the components of the spin primary fields for $L\,\mathrm{Spin}_{16}$ give rise to two (non–commuting) action of $L^{\mathrm{pol}}\mathfrak{e}_8$. Indeed, in this case the highest weights of the spin representations, namely $s_\pm=\frac{1}{2}(\theta_1+\cdots+\theta_{n-1}\pm\theta_n)$ with $n=8$ have squared length 2 and it follows that $s_\pm+\Lambda_R$ are both even lattices which, by inspection coincide with the root lattice of E_8.

5.5. Appendix : intermediate lattices with the extension property.

Let G be simply–laced. We give below the list of intermediate lattices $\Lambda_R\subset\Lambda\subset\Lambda_W$ possessing the *extension property*, *i.e.* a skew–symmetric, bilinear form $\omega\in\mathrm{Hom}(\Lambda\wedge\Lambda,\mathbb{T})$ satisfying $\omega(\alpha,\mu)=(-1)^{\langle\alpha,\mu\rangle}$ whenever $\alpha\in\Lambda_R$. The existence of such a form simplifies the formulae (5.3.1) giving the primary fields whose charges have their weights in Λ. It also removes the sole obstruction to extending the representation of LG on the Hilbert space completion of $\mathcal{S}\otimes\mathbb{C}[\Lambda_W]$ to one of the group of discontinuous loops whose endpoints differ by an element of $Z(G)$ lying in the image of Λ [**TL2**].

Lemma 5.5.1. *If Λ/Λ_R is cyclic of order k then Λ has the extension property if and only if $k\langle\lambda,\lambda\rangle \in 2\mathbb{Z}$ where $\lambda \in \Lambda$ is a generator. In this case, the extension is unique and takes values in $\{\pm 1\}$.*

Proof. If $\omega : \Lambda \wedge \Lambda \to \mathbb{T}$ is an extension then $k\lambda \in \Lambda_R$ and therefore $1 = \omega(k\lambda, \lambda) = (-1)^{k\langle\lambda,\lambda\rangle}$. Conversely, if $k\langle\lambda,\lambda\rangle \in 2\mathbb{Z}$ then the skew-symmetric form $\widetilde{\omega}(\alpha\oplus m\lambda, \beta\oplus n\lambda) = (-1)^{\langle\alpha,\beta\rangle+n\langle\alpha,\lambda\rangle+m\langle\lambda,\beta\rangle}$ defined on $\Lambda_R \oplus \mathbb{Z}\lambda$ descends to one on $\Lambda_R \oplus \mathbb{Z}\lambda/-k\lambda\oplus k\lambda \cong \Lambda$ satisfying the extension property $\diamond$

Proposition 5.5.2. *The following is the complete list of subgroups $\{1\} \neq Z \subset Z(G)$, G simply–laced such that $(2\pi i)^{-1}\exp^{-1}(Z)$ has the extension property.*

(i) *Subgroups of $Z(\mathrm{SU}_{2n})$ of even index and all subgroups of $Z(\mathrm{SU}_{2n+1})$.*
(ii) *The subgroup $Z \subset Z(\mathrm{Spin}_{2n})$ such that $\mathrm{Spin}_{2n}/Z = \mathrm{SO}_{2n}$ and all subgroups of $Z(\mathrm{Spin}_{8n})$.*
(iii) $Z(E_6)$

Proof. We proceed by enumeration according to the Lie theoretic of G. In what follows, θ_i, $i = 1\dots n$ and $\langle\cdot,\cdot\rangle$ are the standard basis and inner product in $\mathbb{R}^n$, I the self-dual lattice $\bigoplus_i \theta_i\mathbb{Z}$ and $I^0 = \{\lambda \in I|\,|\lambda| = \sum_i \lambda_i \in 2\mathbb{Z}\}$.

$\mathbf{SU_n}$
The roots are $\theta_i - \theta_j$, $i \neq j$ and span $\Lambda_R = \{\alpha \in I|\,|\alpha| = 0\}$. The weight $\lambda = \theta_1 - \frac{1}{n}(\theta_1 + \dots + \theta_n)$ does not lie in Λ_R nor do $k\lambda$, $k = 0\dots n-1$ and therefore λ generates $\Lambda_W/\Lambda_R \cong Z(SU_n) = \mathbb{Z}_n$. Thus, for any $m|n$, the index m sublattice of Λ_W is generated by Λ_R and $m\lambda$ and possesses the extension property if, and only if $n/m\|m\lambda\|^2 = m(n-1) \in 2\mathbb{Z}$ and therefore iff n is odd or n and m are even.

$\mathbf{Spin_{2n}},\ \mathbf{n \geq 3}$
The simple roots are $\theta_i - \theta_{i+1}$, $i = 1\dots n-1$ and $\theta_{n-1} + \theta_n$ and span $\Lambda_R = I^0$. The weight lattice is easily seen to be $I + \frac{1}{2}(\theta_1 + \dots + \theta_n)\mathbb{Z}$ with minimal dominant weights given by $v = \theta_1$ and $s_\pm = \frac{1}{2}(\theta_1 + \dots + \theta_{n-1} \pm \theta_n)$. We must distinguish two cases :
n even. Then $2s_\pm = (\theta_1 + \theta_2) + \dots + (\theta_{n-1} \pm \theta_n) = 0 \bmod \Lambda_R$ and $2v = 2\theta_1 = 0 \bmod \Lambda_R$ so that $Z(\mathrm{Spin}_{2n}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and each of the minimal weights generates a subgroup of order 2. In SO_{2n}, $\exp_T(-2\pi i v) = 1$ and therefore $\mathbb{Z}_2^v$ possesses the extension property since $2\|v\|^2 = 2$. The groups $\mathbb{Z}_2^{s_\pm}$ have it if and only if n is a multiple of 4 since $2\|s_\pm\|^2 = \frac{n}{2}$. Lastly, if $Z(\mathrm{Spin}_{2n})$ has the extension property then so do $\mathbb{Z}_2^{s_\pm}$ and therefore n is a multiple of 4. Conversely, if that is the case, the skew-symmetric form $\omega(\alpha \oplus pv \oplus qs_+, \beta \oplus p'v \oplus q's_+) = i^{pq'-qp'}(-1)^{\langle\alpha,\beta\rangle+\langle\alpha,p'v+q's_+\rangle+\langle pv+qs_+,\beta\rangle}$ defined on $\Lambda_R \oplus v\mathbb{Z} \oplus s_+\mathbb{Z}$ descends to one on $\Lambda_W = \Lambda_R \oplus v\mathbb{Z} \oplus s_+\mathbb{Z}/(-2v \oplus 2v \oplus 0)\mathbb{Z} + (-2s_+ \oplus 0 \oplus 2s_+)\mathbb{Z}$. In this case the extension is unique only up to multiplication by the non-trivial $\mathbb{Z}_2$-valued skew-form on $\mathbb{Z}_2 \times \mathbb{Z}_2$.
n odd. Then $2s_\pm = \theta_1 + (\theta_2 + \theta_3) + \dots + (\theta_{n-1} \pm \theta_n) = v \bmod \Lambda_R$ and therefore $Z(\mathrm{Spin}_{2n}) \cong \mathbb{Z}_4$ with v of order 2 and $s_\pm$ of order 4. Therefore $\Lambda_R + v$ possesses the extension property because $2\|v\|^2 = 2$ but $Z(\mathrm{Spin}_{2n})$ doesn't for $4\|s_+\|^2 = n \in 2\mathbb{Z}+1$.

$\mathbf{E_6}$
The simple roots of E_6 are $\alpha_1 = \frac{1}{2}(\theta_1 - \theta_2 - \dots - \theta_7 + \theta_8), \alpha_2 = \theta_2 + \theta_1, \alpha_3 = \theta_2 - \theta_1, \dots, \alpha_6 = \theta_5 - \theta_4$. The fundamental weight $\lambda_1 = -\frac{2}{3}(\theta_6 + \theta_7 - \theta_8)$ does not lie in the root lattice and therefore generates $\Lambda_W/\Lambda_R \cong \mathbb{Z}_3$ [**Bou**, Planche V]. Moreover, $3\|\lambda_1\|^2 = 4$.

$\mathbf{E_7}$
The simple roots of E_7 are $\alpha_1 = \frac{1}{2}(\theta_1 - \theta_2 - \dots - \theta_7 + \theta_8), \alpha_2 = \theta_2 + \theta_1, \alpha_3 = \theta_2 - \theta_1, \dots, \alpha_7 = \theta_6 - \theta_5$. The fundamental weight $\lambda_2 = \frac{1}{2}(\theta_1 + \dots + \theta_6 - 2\theta_7 + 2\theta_8)$ does not lie in the root lattice and therefore generates $\Lambda_W/\Lambda_R \cong \mathbb{Z}_2$ [**Bou**, Planche VI]. Moreover, $2\|\lambda_2\|^2 = 5$.

$\mathbf{E_8}$
E_8 has trivial centre [**Bou**, Planches VII] and therefore no intermediate lattices $\diamond$

Remark. Notice that the above list agrees with the isomorphisms $\mathrm{Spin}_6 \cong SU_4$.

CHAPTER VI

Analytic properties of primary fields

This chapter is devoted to the study of the continuity properties of primary fields of $L\,\mathrm{Spin}_{2n}$. These are required to construct (at first unbounded) explicit intertwiners for the local loop groups by smearing the fields on functions supported in complementary intervals. We show that any primary field $\phi:\mathcal{H}_i^{\mathrm{fin}}\otimes V_k[z,z^{-1}]\rightarrow\mathcal{H}_j^{\mathrm{fin}}$ such that one of the Spin_{2n}–modules $\mathcal{H}_i(0),V_k,\mathcal{H}_j(0)$ is minimal extends to a jointly continuous operator–valued distribution $\mathcal{H}_i^\infty\otimes C^\infty(S^1,V_k)\rightarrow\mathcal{H}_j^\infty$ satisfying as expected

$$\pi_j(\gamma)\phi(f)\pi_i(\gamma)^*=\phi(\gamma f) \tag{1}$$

for any $\gamma\in L\,\mathrm{Spin}_{2n}$. When the charge V_k is the vector representation, ϕ satisfies stronger continuity properties and extends to a bounded map $L^2(S^1,V_k)\rightarrow\mathcal{B}(\mathcal{H}_i,\mathcal{H}_j)$.

The level 1 result is obtained in section 1 from the bosonic construction of the primary fields given in chapter V. The level ℓ result is proved in section 3 and follows because the primary fields belonging to the above class may be obtained as ℓ–fold tensor products of level 1 primary fields. Some care is required in checking this and the corresponding finite–dimensional analysis is carried out in section 2. The identity (1) is proved in section 4.

1. Continuity of level 1 primary fields

As observed by Wassermann [**Wa5**], the continuity of the level 1 primary fields depends upon the fact that they may be written as the product of generating functions whose modes are bounded operators. We begin by studying these. The notation follows chapter V.

1.1. Norm boundedness of the vertex operators for $T\times\check{T}$.

Recall that the action of $\mathfrak{t}_\mathbb{C}$ on $\mathbb{C}[\Lambda_W]$ is given by

$$h\delta_\mu=\langle h,\mu\rangle\delta_\mu \tag{1.1.1}$$

LEMMA 1.1.1. *Let η be a $\mathbb{T}$–valued function on Λ_W and*

$$X_\lambda(z)=V_\lambda z^{\lambda+\frac{\langle\lambda,\lambda\rangle}{2}}\eta(\cdot)=\sum_m X_\lambda(m)z^{-m} \tag{1.1.2}$$

Then, $X_\lambda(m)$ is bounded in norm by one.

PROOF. By (1.1.1), $X_\lambda(m)=V_\lambda\eta(\cdot)P_m$ where P_m is the orthogonal projection onto

$$\bigoplus_{\substack{\mu\in\Lambda_W\\ \langle\mu,\lambda\rangle+\frac{\langle\lambda,\lambda\rangle}{2}=-m}}\mathbb{C}\cdot\delta_\mu \tag{1.1.3}$$

The claimed boundedness follows since $V_\lambda\eta(\cdot)$ is unitary $\diamond$

1.2. Norm boundedness of pre-vertex operators of small conformal dimension.

This subsection follows [**Wa5**]. Recall from §3.2 of chapter V the definition of the exponential operators

$$E^-(\alpha,z)=\exp\Big(-\sum_{n<0}\frac{\alpha(n)}{n}z^{-n}\Big)\qquad\qquad E^+(\alpha,z)=\exp\Big(-\sum_{n>0}\frac{\alpha(n)}{n}z^{-n}\Big) \tag{1.2.1}$$

which are formal power series with coefficents in $\mathrm{End}(\mathcal{S})$ and the fact that

$$E^+(\alpha,z)E^-(\beta,\zeta)=\Big(1-\frac{\zeta}{z}\Big)^{\langle\alpha,\beta\rangle}E^-(\beta,\zeta)E^+(\alpha,z) \tag{1.2.2}$$

We have now

Proposition 1.2.1. *For any $\alpha\in i\mathfrak{t}\cong\mathbb{R}^n$ such that $\langle\alpha,\alpha\rangle\leq 1$, the modes $Y(n)$ of the pre–vertex operator*

$$Y_\alpha(z)=E^-(\alpha,z)E^+(\alpha,z)=\exp\Big(-\sum_{n<0}\frac{\alpha(n)}{n}z^{-n}\Big)\exp\Big(-\sum_{n>0}\frac{\alpha(n)}{n}z^{-n}\Big) \tag{1.2.3}$$

satisfy

$$\|Y(n)\psi\|\leq\|\psi\| \tag{1.2.4}$$

for any $\mathbb{Z}$ and $\psi\in\mathcal{S}$.

Proof. Consider first the case $\langle\alpha,\alpha\rangle=1$. Then,

$$Y_\alpha(z)Y_{-\alpha}(\zeta)+\Big(\frac{\zeta}{z}\Big)^{-1}Y_{-\alpha}(\zeta)Y_\alpha(z)=\Big[\Big(1-\frac{\zeta}{z}\Big)^{-1}+\Big(1-\frac{z}{\zeta}\Big)^{-1}\Big(\frac{\zeta}{z}\Big)^{-1}\Big]\mathcal{R}(\alpha,z,\zeta) \tag{1.2.5}$$

where

$$\mathcal{R}(\alpha,z,\zeta)=E^-(\alpha,z)E^-(-\alpha,\zeta)E^+(\alpha,z)E^+(-\alpha,\zeta)=\mathcal{R}(-\alpha,\zeta,z) \tag{1.2.6}$$

By (V.1.8), the bracketed term is equal to $\delta\Big(\frac{\zeta}{z}\Big)$ and since $\mathcal{R}(\alpha,\zeta,\zeta)=1$, we find by (V.1.10)

$$Y_\alpha(z)Y_{-\alpha}(\zeta)+\Big(\frac{\zeta}{z}\Big)^{-1}Y_{-\alpha}(\zeta)Y_\alpha(z)=\delta\Big(\frac{\zeta}{z}\Big) \tag{1.2.7}$$

Writing $Y_\alpha(z)=\sum_n a_nz^{-n}$, $Y_{-\alpha}(\zeta)=\sum_n b_nz^{-n}$ and recalling that the formal adjunction property $Y_\alpha(z)^*=Y_{-\alpha}(z)$ implies that $b_n^*=a_{-n}$, we get by taking modes on both sides

$$a_na_m^*+a_{m-1}^*a_{n-1}=\delta_{n,m} \tag{1.2.8}$$

and in particular

$$\|a_n\psi\|\leq\|\psi\| \tag{1.2.9}$$

The general case may be settled by the following factorisation trick. The space $V_+=z^{-1}\mathfrak{t}_\mathbb{C}[z^{-1}]$ splits as $V_+^1\bigoplus V_+^2$ for any orthogonal decomposition $\mathfrak{t}_\mathbb{C}=\mathfrak{t}_\mathbb{C}^1\oplus\mathfrak{t}_\mathbb{C}^2$. Correspondingly,

$$\mathcal{S}=\bigoplus_k S^kV_+=\mathcal{S}^1\bigotimes\mathcal{S}^2 \tag{1.2.10}$$

and

$$Y_{\lambda\oplus\lambda'}(z)=Y_\lambda(z)\otimes Y_{\lambda'}(z) \tag{1.2.11}$$

for any $\lambda\in\mathfrak{t}_\mathbb{C}^1$, $\lambda'\in\mathfrak{t}_\mathbb{C}^2$. If $\langle\alpha,\alpha\rangle\leq 1$, we may find, by possibly enlarging $\mathfrak{t}_\mathbb{C}$ if it is one–dimensional, $i\mathfrak{t}\ni\alpha'\perp\alpha$ such that $\langle\alpha\oplus\alpha',\alpha\oplus\alpha'\rangle=1$. The modes $Y_{\alpha\oplus\alpha'}(n)$ satisfy (1.2.4) and are equal to

$$\sum_{p+q=n}Y_\alpha(p)\otimes Y_{\alpha'}(q) \tag{1.2.12}$$

Let $d=d_1+d_2$ be the infinitesimal generators of rotations on $\mathcal{S}^1,\mathcal{S}^2$. Let $\eta\in\mathcal{S}^2$ be the lowest energy vector so that $Y_{\alpha'}(z)\eta=E^-(\alpha',z)\eta$ and therefore $Y_{\alpha'}(0)\eta=\eta$. If $\xi\in\mathcal{S}^1$ is any eigenvector of d_1, then

$$\|\xi\|^2\|\eta\|^2=\|\xi\otimes\eta\|^2\geq\|Y_{\alpha\oplus\alpha'}(n)\xi\otimes\eta\|^2=\sum_{p'+q'=n}\|Y_\alpha(p')\xi\|^2\|Y_{\alpha'}(q')\eta\|^2\geq\|Y_\alpha(n)\xi\|^2\|\eta\|^2 \tag{1.2.13}$$

since the vectors $Y_\alpha(p)\xi$, $Y_\alpha(p')\xi$ have different energies and are therefore orthogonal for $p\neq p'$ and the same holds for $Y_\alpha(q)\eta$ and $Y_\alpha(q')\xi$ whenever $q\neq q'$. Thus

$$\|Y_\alpha(n)\xi\|\leq\|\xi\| \tag{1.2.14}$$

Lastly, if $\xi=\sum\xi_n$ is a sum of eigenvectors of d_1 with distinct eigenvalues, then

$$\|Y_\alpha(m)\xi\|^2=\sum\|Y_\alpha(m)\xi_n\|^2\leq\sum\|\xi_n\|^2=\|\xi\|^2 \tag{1.2.15}$$

◇

1.3. Sobolev estimates for products of norm bounded homogeneous fields.

This subsection follows [**Wa5**]. Let $\mathcal{F}_i$, $i = 1 \dots k$ be inner product spaces supporting positive energy representations $U^i_\theta = e^{i\theta d_i}$ of (a cover of) $\mathrm{Rot}\, S^1$. Call a field $Y_i(z) = \sum_n Y_i(n) z^{-n} \in \mathrm{End}(\mathcal{F}_i)[[z, z^{-1}]]$ *homogeneous* if $[d_i, Y_i(n)] = -nY_i(n)$.

Proposition 1.3.1. *Let $Y_1(z) \cdots Y_k(z)$ be homogeneous fields with uniformly bounded modes acting on $\mathcal{F}_1 \cdots \mathcal{F}_k$. Then, the modes of $Y(z) = Y_1(z) \otimes \cdots \otimes Y_k(z)$ satisfy*

$$\|Y(m)\xi\| \leq C(1 + |m|)^{k-1} \|(1 + d)^{k-1}\xi\| \tag{1.3.1}$$

for any $\xi \in \mathcal{F} = \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_k$ where $d = d_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes d_k$

Proof. We have

$$Y(m) = \sum_{p_1+\cdots+p_k=m} Y_1(p_1) \otimes \cdots \otimes Y_k(p_k) \tag{1.3.2}$$

Assume the $\mathcal{F}_i$ support positive energy representations of an s–sheeted cover of $\mathrm{Rot}\, S^1$. If the right hand–side is applied to $\xi \in \mathcal{F}$ of energy $n \geq 0$, the sum reduces to one involving only terms for which $p_i \leq n$ for any i. Their number is bounded by that of solutions of $\sum p_i = m$, $p_i \leq n$, $p_i \in s^{-1}\mathbb{Z}$. However, $p_i = m - \sum_{j \neq i} p_j \geq m - n(k - 1)$ and hence for any i,

$$m - n(k - 1) \leq p_i \leq n \tag{1.3.3}$$

so that there are at most $s^{k-1}(1+nk-m)^{k-1} \leq s^{k-1}(1+nk+|m|)^{k-1}$ solutions since p_k is determined once $p_1, \dots p_{k-1}$ are fixed. It follows from $\|A \otimes B\| \leq \|A\|\|B\|$ for operators A, B (Cauchy–Schwarz) that

$$\begin{aligned} \|Y(m)\xi\| &\leq Cs^{k-1}(1 + nk + m)^{k-1}\|\xi\| \\ &\leq C'(1 + |m|)^{k-1}(1 + n)^{k-1}\|\xi\| \\ &= C'(1 + |m|)^{k-1}\|(1 + d)^{k-1}\xi\| \end{aligned} \tag{1.3.4}$$

where C, C' are constants independent of m and ξ and we have used $(1+d)\xi = (1+n)\xi$. The claimed inequality therefore holds for eigenvectors of d and hence for any $\xi \in \mathcal{F}$ since it is stable under taking orthogonal sums of eigenvectors $\diamond$

1.4. Continuity of the level 1 spin primary fields.

Theorem 1.4.1. *Let $\phi_s : \mathcal{H}_i^{\mathrm{fin}} \otimes V_s[z, z^{-1}] \to \mathcal{H}_j^{\mathrm{fin}}$ be a level 1 primary field of $L\,\mathrm{Spin}_{2n}$ whose charge is one of the spin modules. Then ϕ_s extend to a jointly continuous map $\mathcal{H}_i^\infty \otimes C^\infty(S^1, V_s) \to \mathcal{H}_j^\infty$.*

Proof. Denote as customary by $\mathcal{H}_i^t, \mathcal{H}_j^t$ the completion of $\mathcal{H}_i^{\mathrm{fin}}, \mathcal{H}_j^{\mathrm{fin}}$ with respect to the norm $\|\xi\|_t = \|(1+d)^t\xi\|$ where d is the infinitesimal generator of rotations. The weights of the spin representations are of the form $\mu = \frac{1}{2}(\epsilon_1\theta_1 + \cdots + \epsilon_n\theta_n)$ where $\epsilon_i \in \{\pm 1\}$ and may therefore be decomposed as orthogonal sums of $\Delta = \lceil \frac{n}{4} \rceil$ vectors λ_i with $\langle \lambda_i, \lambda_i \rangle \leq 1$. By theorem V.5.3.2, the corresponding component of ϕ_s factorises as

$$\Phi_\mu(z) = Y_{\lambda_1}(z) \otimes \cdots \otimes Y_{\lambda_\Delta}(z) \otimes V_\mu z^{\mu + \frac{\langle \mu, \mu \rangle}{2}} \eta(\cdot) \tag{1.4.1}$$

for some $\mathbb{T}$–valued function η on Λ_W. By lemma 1.1.1 and proposition 1.2.1, all factors have modes bounded in norm by 1. Thus, if $\xi \in \mathcal{H}_i^{\mathrm{fin}}$ is of energy $n \geq m$, proposition 1.3.1 yields

$$\begin{aligned} \|\Phi_\mu(m)\xi\|_t &= (1 + n - m)^t \|\Phi_\mu(m)\xi\| \\ &\leq C(1 + |m|)^\Delta (1 + n - m)^t \|\xi\|_\Delta \\ &= C(1 + |m|)^\Delta \frac{(1 + n - m)^t}{(1 + n)^t} \|\xi\|_{\Delta+t} \\ &\leq C(1 + |m|)^{\Delta + \frac{|t|}{2}} \|\xi\|_{\Delta+t} \end{aligned} \tag{1.4.2}$$

whence, for any $\xi\in\mathcal{H}_i^{\text{fin}}$

$$\|\Phi_\mu(m)\xi\|_t\leq C(1+|m|)^{\Delta+|t|}\|\xi\|_{\Delta+t} \tag{1.4.3}$$

since ξ may be written as an orthogonal sum of eigenvectors of d. Next, if

$$f=\sum_{\substack{m\in\mathbb{Z}\\ \mu\in\Pi(s)}} a_{m,\mu}v_\mu(m)\in V_s[z,z^{-1}] \tag{1.4.4}$$

where $\Pi(s)$ is the set of weights of V_s, we have, by definition $\phi_s(f)=\sum_{m,\mu}a_{m,\mu}\Phi_\mu(m)$. Then,

$$\|\phi_s(f)\xi\|_t\leq C\sum_{m,\mu}|a_{m,\mu}|(1+|m|)^{\Delta+|t|}\|\xi\|_{\Delta+t}\leq C'|f|_{\Delta+|t|}\|\xi\|_{\Delta+t} \tag{1.4.5}$$

and ϕ_s extends to a continuous map $C^\infty(S^1,V_s)\rightarrow\mathcal{B}(\mathcal{H}_i^{\Delta+t},\mathcal{H}_j^t)$ $\diamond$

Remark. Notice that the estimates (1.4.5) are not quite optimal. For example, for $L\,\mathrm{Spin}_8$, the spin primary fields are Fermi fields by proposition V.5.4.2 and therefore extend to bounded maps $L^2(S^1,V_{s_\pm})\rightarrow\mathcal{B}(\mathcal{H}_i,\mathcal{H}_j)$.

2. Finite–dimensional Spin_{2n}–intertwiners

By lemma I.2.3.1, any Spin_{2n}–module admissible at level ℓ is contained in an ℓ–fold tensor product $V_{p_1}\otimes\cdots\otimes V_{p_\ell}$ where the V_p are admissible at level 1 and therefore minimal. In this section, we prove an analogous factorisation for intertwiners $\phi:V_i\otimes V_k\rightarrow V_j$ when $V_i,V_{k,j}$ are admissible at level ℓ and one of them is minimal. This will be used in the next section to show that level ℓ primary fields corresponding to the vertex $\begin{pmatrix} V_k \\ V_j\ V_i\end{pmatrix}$ can be written as the tensor product of level 1 primary fields.

2.1. Minimal representations.

Recall from proposition I.2.2.1 that the weights of a minimal G–module V lie on a single orbit of the Weyl group and satisfy $\langle\mu,\alpha^\vee\rangle\in\{1,0,-1\}$ for any root α and corresponding coroot $\alpha^\vee=h_\alpha$. If $\mu,\widetilde{\mu}$ are weights of V, then $\widetilde{\mu}=\mu+\alpha_1+\cdots+\alpha_k$ where the α_i are possibly repeated roots. Since $\langle\mu,\alpha_i\rangle\geq 0$ for all i would imply $\|\widetilde{\mu}\|>\|\mu\|$ there exists an α_i such that $\langle\mu,\alpha_i^\vee\rangle=-1$ and therefore $\|\mu+\alpha_i\|=\|\mu\|$. An iteration of this argument produces a permutation τ of $\{1,\ldots,k\}$ such that $\|\mu+\alpha_{\tau(1)}+\cdots+\alpha_{\tau(j)}\|=\|\mu\|$ for any $j=1\ldots k$. In representation theoretic terms, the weight spaces of a minimal representation are one–dimensional and if $v_{\widetilde{\mu}}$, v_μ are eigenvectors corresponding to the weights $\widetilde{\mu}$, μ then, up to a non–zero multiplicative constant, $v_{\widetilde{\mu}}=e_{\alpha_{\tau(k)}}\cdots e_{\alpha_{\tau(1)}}v_\mu$. Indeed, by elementary $\mathfrak{sl}_2(\mathbb{C})$ theory, for any $j\in\{1\ldots k\}$, $e_{\alpha_{\tau(j)}}e_{\alpha_{\tau(j-1)}}\cdots e_{\alpha_{\tau(1)}}v_\mu\neq 0$ since $\langle\alpha_j^\vee,\mu+\alpha_1+\cdots+\alpha_{j-1}\rangle=-1$ and therefore $v_{\widetilde{\mu}}$ and $e_{\alpha_{\tau(k)}}\cdots e_{\alpha_{\tau(1)}}v_\mu$ are proportional since they lie in the same weight space.

If V_δ,V_ν are irreducible representations of highest weights δ,ν and V_δ is minimal, the tensor product $V_\nu\otimes V_\delta$ decomposes according to proposition I.2.2.2 as

$$V_\nu\otimes V_\delta=\bigoplus_{\widetilde{\delta}}V_{\nu+\widetilde{\delta}} \tag{2.1.1}$$

where $\widetilde{\delta}$ varies among the weights of V_δ such that $\nu+\widetilde{\delta}$ is dominant.

Lemma 2.1.1. *Let V_ν, V_δ be irreducible representations with highest weights ν, δ and corresponding eigenvectors v_ν, v_δ. Assume V_δ is minimal so that $V_\nu\otimes V_\delta$ decomposes according to* (2.1.1). *Then, if $\widetilde{\delta}$ is a weight of V_δ such that $\nu+\widetilde{\delta}$ is dominant and $v_{\widetilde{\delta}}$ is a corresponding non–zero weight vector, the orthogonal projection of $v_\nu\otimes v_{\widetilde{\delta}}$ on $V_{\nu+\widetilde{\delta}}\subset V_\nu\otimes V_{\widetilde{\delta}}$ is non–zero.*

Proof. Let $\Omega_{\nu+\widetilde{\delta}}\in V_{\nu+\widetilde{\delta}}\subset V_\nu\otimes V_{\widetilde{\delta}}$ be a non–zero highest weight vector. We claim that $(\Omega_{\nu+\widetilde{\delta}},v_\nu\otimes v_{\widetilde{\delta}})\neq 0$. Indeed, if the contrary holds, we shall prove inductively that

$$(\Omega_{\nu+\widetilde{\delta}},f_{\alpha_{\sigma(k)}}\cdots f_{\alpha_{\sigma(1)}}v_\nu\otimes e_{\alpha_k}\cdots e_{\alpha_1}v_{\widetilde{\delta}})=0 \tag{2.1.2}$$

where $\alpha_1 \dots \alpha_k$ are (possibly repeated) simple roots and σ is any permutation of $\{1, \dots, k\}$. Since such vectors form a spanning set of the eigenspace of $V_\nu \otimes V_\delta$ corresponding to the weight $\nu + \widetilde{\delta}$, it follows that $\Omega_{\nu+\widetilde{\delta}} = 0$, a contradiction. To prove the inductive claim, notice that

$$\begin{aligned} &f_{\alpha_{\sigma(k)}}(f_{\alpha_{\sigma(k-1)}} \cdots f_{\alpha_{\sigma(1)}} v_\nu \otimes e_{\alpha_k} \cdots e_{\alpha_1} v_{\widetilde{\delta}}) \\ =&f_{\alpha_{\sigma(k)}} \cdots f_{\alpha_{\sigma(1)}} v_\nu \otimes e_{\alpha_k} \cdots e_{\alpha_1} v_{\widetilde{\delta}} \\ -&\langle \alpha_k^\vee, \widetilde{\delta} + (\alpha_1 + \cdots + \alpha_{j-1}) \rangle f_{\alpha_{\sigma(k-1)}} \cdots f_{\alpha_{\sigma(1)}} v_\nu \otimes e_{\alpha_k} \cdots \widehat{e_{\alpha_j}} \cdots e_{\alpha_1} v_{\widetilde{\delta}} \\ +&f_{\alpha_{\sigma(k-1)}} \cdots f_{\alpha_{\sigma(1)}} v_\nu \otimes e_{\alpha_k} \cdots e_{\alpha_j} f_{\alpha_{\sigma_k}} e_{\alpha_{j-1}} \cdots e_{\alpha_1} v_{\widetilde{\delta}} \end{aligned} \tag{2.1.3}$$

where j is the largest element of $\{1, \dots, k\}$ such that $\alpha_j = \alpha_{\sigma(k)}$. Now, if $f_{\alpha_{\sigma(k)}} \cdots f_{\alpha_{\sigma(1)}} v_\nu \otimes e_{\alpha_k} \cdots e_{\alpha_1} v_{\widetilde{\delta}}$ is non–zero, then $e_{\alpha_j} e_{\alpha_{j-1}} \cdots e_{\alpha_1} v_{\widetilde{\delta}} \neq 0$ and therefore by elementary $\mathfrak{sl}_2(\mathbb{C})$ theory, $f_{\alpha_{\sigma_k}} e_{\alpha_{j-1}} \cdots e_{\alpha_1} v_{\widetilde{\delta}} = 0$ since $\widetilde{\delta} + (\alpha_1 + \cdots + \alpha_{j-1})$ is a minimal weight and therefore $\langle \alpha_j^\vee, \widetilde{\delta} + (\alpha_1 + \cdots + \alpha_{j-1}) \rangle \in \{1, 0, -1\}$. Therefore, if $f_{\alpha_{\sigma(k)}} \cdots f_{\alpha_{\sigma(1)}} v_\nu \otimes e_{\alpha_k} \cdots e_{\alpha_1} v_{\widetilde{\delta}} \neq 0$, the last term on the right hand-side of (2.1.3) vanishes and by the inductive hypothesis and the fact that $\Omega_{\nu+\widetilde{\delta}}$ is a highest weight vector, (2.1.2) holds $\diamond$

2.2. Admissible intertwiners for Spin_{2n}.

We begin by collecting some elementary facts about the spin representations of Spin_{2n} which may be found in section III.1. The complexified definining representation $V = V_{\square} = \mathbb{C}^{2n}$ has a natural bilinear, non–degenerate symmetric form $B(\cdot, \cdot)$ which yields an isomorphism $\mathfrak{so}_{2n,\mathbb{C}} \cong V \wedge V$ where the latter space acts on V by $u \wedge v\, w = uB(v, w) - v(u, w)$. If f_j, $0 \neq j = -n \dots n$ is an orthonormal basis of V satisfying $B(f_j, f_k) = \delta_{j+k,0}$, a basis for $\mathfrak{so}_{2n,\mathbb{C}}$ is given by the elementary matrices $F_{ij} = f_i \wedge f_j$. The Cartan subalgebra corresponding to the block diagonal matrices in $\mathfrak{so}_{2n,\mathbb{C}}$ is then spanned by $F_{i,-i}$, $i = 1, \dots n$. The roots of Spin_{2n} are $\theta_k + \theta_l$, $-n \leq k \neq \pm l \leq n$ where the θ_i, $i = 1 \dots n$ are the dual basis to $F_{i,-i}$ and, by definition $\theta_{-i} = -\theta_i$. The $\mathfrak{sl}_2(\mathbb{C})$–subalgebra $\{e_\alpha, f_\alpha, h_\alpha\}$ of $\mathfrak{so}_{2n,\mathbb{C}}$ corresponding to $\alpha = \theta_k + \theta_l$ is given by $\{F_{k,l}, F_{k,-k} + F_{l,-l}, F_{-l,-k}\}$. The weight vectors in V are exactly the f_j since $F_{i,-i} f_j = (\delta_{ij} - \delta_{i,-j}) f_i = \theta_j(F_{i,-i}) f_j$.

The spin representations are obtained via the representation of the Clifford algebra of V generated by the $\mathbb{C}$-linear symbols $\psi(u)$, $u \in V$ subject to the relations $\psi(u)\psi(v) + \psi(v)\psi(u) = 2B(u, v)$, on the exterior algebra $\Lambda V^{1,0}$ where $V^{1,0}$ is the subspace spanned by the f_j, $j > 0$. The action is given explicitly by

$$\psi(u) v_1 \wedge \cdots \wedge v_k = \begin{cases} \sqrt{2} u \wedge v_1 \wedge \cdots \wedge v_k & \text{if } u \in V^{1,0} \\ \sqrt{2} \sum_j (-1)^{j+1} B(u, v_j) v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k & \text{if } u \in V^{0,1} \end{cases} \tag{2.2.1}$$

The representation is obtained by letting the Lie algebra element $u \wedge v$ act as $\frac{1}{4}(\psi(u)\psi(v) - \psi(v)\psi(u)) = \frac{1}{2}(\psi(u)\psi(v) - B(u, v))$. The even and odd parts of the exterior algebra are clearly invariant under this action and are irreducible. Their weights are easily read from

$$F_{i,-i} f_{j_1} \wedge \cdots \wedge f_{j_k} = (\psi(f_i)\psi(f_{-i}) - \frac{1}{2}) f_{j_1} \wedge \cdots \wedge f_{j_k} = \begin{cases} \frac{1}{2} f_{j_1} \wedge \cdots \wedge f_{j_k} & \text{if } i \in \{j_1, \dots, j_k\} \\ -\frac{1}{2} f_{j_1} \wedge \cdots \wedge f_{j_k} & \text{if } i \notin \{j_1, \dots, j_k\} \end{cases} \tag{2.2.2}$$

so that the vector $f_J = f_{j_1} \wedge \cdots \wedge f_{j_k}$ corresponds to the weight $-\frac{1}{2} \sum \theta_i + \sum_p \theta_{j_p}$. The highest weight of the half of the exterior algebra containing the top exterior power $\Lambda^n V^{1,0}$ has therefore highest weight $s_+ = \frac{1}{2}(\theta_1 + \cdots + \theta_n)$ and the other has highest weight $s_- = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} - \theta_n)$.

Finally, the Clifford multiplication map $V \otimes \Lambda V^{1,0} \to \Lambda V^{1,0}$ given by $v \otimes f_J \to \psi(v) f_J$ commutes with the action of Spin_{2n} and gives therefore rise to two intertwiners $V \otimes V_{s_\pm} \to V_{s_\mp}$.

LEMMA 2.2.1. *Let V_i, V_j, V_k be irreducible representations of $G = \mathrm{Spin}_{2n}$, one of which is minimal so that* $\mathrm{Hom}_G(V_i \otimes V_k, V_j)$ *is at most one–dimensional. If all are admissible at level ℓ and* $\mathrm{Hom}_G(V_i \otimes$

$V_k, V_j) = \mathbb{C}$, there exist minimal G–modules $V_{i_p}, V_{j_p}, V_{k_p}$, $p = 1 \dots \ell$ and intertwiners $\phi_p \in \mathrm{Hom}_G(V_{i_p} \otimes V_{k_p}, V_{j_p})$ such that

$$V_i \subset \bigotimes_{p=1}^{\ell} V_{i_p} \qquad V_j \subset \bigotimes_{p=1}^{\ell} V_{j_p} \qquad V_k \subset \bigotimes_{p=1}^{\ell} V_{k_p} \tag{2.2.3}$$

and the corresponding restriction of $\otimes_p \phi_p$ to an intertwiner $V_i \otimes V_k \to V_j$ is non–zero. Moreover, if V_k is minimal then one may choose $V_{i_p} = V_{j_p}$ and $V_{k_p} = \mathbb{C}$ for $p = 1 \dots \ell - 1$ and $V_{k_\ell} = V_k$ so that $\otimes_p \phi_p$ is of the form $1 \otimes \cdots \otimes 1 \otimes \phi_p$.

Proof. Up to a permutation of the modules, we may assume that V_k is minimal and that $\langle \mu, \theta \rangle \leq \langle \lambda, \theta \rangle$ where μ, λ are the highest weights of V_i and V_j respectively and θ is the highest root. If $\langle \mu, \theta \rangle \leq \ell - 1$, V_i is contained, by lemma I.2.3.1 in some tensor product $V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}}$ with minimal factors. Then,

$$V_i \subset V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}} \otimes \mathbb{C} \tag{2.2.4}$$

$$V_k \subset \mathbb{C} \otimes \cdots \otimes \mathbb{C} \otimes V_k \tag{2.2.5}$$

$$V_j \subset V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}} \otimes V_k \tag{2.2.6}$$

and $\phi = 1 \otimes \cdots \otimes 1 \otimes 1$ clearly restricts to a non–zero intertwiner. Assume now that $\langle \mu, \theta \rangle = \langle \lambda, \theta \rangle = \ell \geq 2$. If V_k is the trivial representation then $V_i = V_j$ and, by lemma I.2.3.1, $V_i \subset V_{i_1} \otimes \cdots \otimes V_{i_\ell}$ for some minimal V_{i_p} and the lemma holds. We shall treat the cases when V_k is the vector or one of the spin representations separately.

$\mathbf{V_k = V_{s_\pm}}$.
Let s and $\overline{s}$ be the highest weights of V_k and of the other spin representation so that $s = s_\pm = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} \pm \theta_n)$ and $\overline{s} = s_\mp$. We have $\lambda = \mu + \sigma$ where $\sigma = \frac{1}{2}(\epsilon_1\theta_1 + \cdots + \epsilon_n\theta_n)$ for some $\epsilon_i \in \{\pm 1\}$ is a weight of $V_k = V_s$. Up to a permutation of V_i, V_j, we may assume that $\mu_1 = \lambda_1 + \frac{1}{2}$. Since $\lambda_1 + \lambda_2 = \langle \lambda, \theta \rangle = \langle \mu, \theta \rangle = \mu_1 + \mu_2$, we have $\mu_2 = \lambda_2 - \frac{1}{2}$ and therefore $\mu_1 - \mu_2 = \lambda_1 - \lambda_2 + 1 \geq 1$ so that $\rho = \mu - \theta_1$ is a dominant weight. Let V_ρ be the corresponding irreducible representation. Since $\langle \rho, \theta \rangle = \ell - 1$, V_ρ is contained in some tensor product $V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}}$ with minimal factors. Thus, by (2.1.1)

$$V_i \subset V_\rho \otimes V_\square \subset V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}} \otimes V_\square \tag{2.2.7}$$

$$V_k \subset \mathbb{C} \otimes \cdots \otimes \mathbb{C} \otimes V_s \tag{2.2.8}$$

$$V_j \subset V_\rho \otimes V_{\overline{s}} \subset V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}} \otimes V_{\overline{s}} \tag{2.2.9}$$

since $\mu = \rho + \theta_1$ and $\lambda = \mu + \sigma = \rho + \sigma'$ where $\sigma' = \theta_1 + \sigma$ is a weight of $V_{\overline{s}}$. If $\psi : V_\square \otimes V_s \to V_{\overline{s}}$ is Clifford multiplication, we claim that the intertwiner $\phi = 1 \otimes \cdots \otimes 1 \otimes \psi$ has a non–zero restriction to $V_i \otimes V_k \to V_j$. To see this, notice that the highest weight vector v_μ in $V_i \subset V_\rho \otimes V_\square$ is the product $v_\rho \otimes v_{\theta_1} = v_\rho \otimes f_1$ of the corresponding highest weight vectors. If $v_\sigma \in V_s$ is of weight σ so that $v_\sigma = \wedge_{j:\sigma_j = \frac{1}{2}} f_j$, then $\phi(v_\mu \otimes v_\sigma) = v_\rho \otimes \psi(f_1 \otimes v_\sigma) = v_\rho \otimes f_1 \wedge v_\sigma$. Since $f_1 \wedge v_\sigma \in V_{\overline{s}}$ is of weight $\sigma + \theta_1$ and $\lambda = \rho + (\sigma + \theta_1)$, lemma 2.1.1 implies that $\phi(v_\mu \otimes v_\sigma)$ has a non–zero projection on $V_j \subset V_\rho \otimes V_{\overline{s}}$ whence the conclusion.

$\mathbf{V_k = V_\square}$.
Up to a permutation of V_i and V_j, we may assume that λ is obtained from μ by adding a box to the corresponding Young diagram, *i.e.* $\lambda = \mu + \theta_j$ where $j \geq 3$ since $\langle \lambda, \theta \rangle = \langle \mu, \theta \rangle$. Let

$$\sigma = \frac{1}{2}(\theta_1 + \cdots + \theta_{j-1} - \theta_j - \cdots - \theta_n) \tag{2.2.10}$$

so that it is a weight of V_s with $s = s_\pm$ according to the parity of $n - j + 1$. As is readily verified, $\rho = \mu - \sigma$ is dominant and satisfies $\langle \rho, \theta \rangle = \ell - 1$. Thus, if V_ρ is the corresponding highest weight

representation, then $V_\rho \subset V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}}$ where the V_{i_p} are minimal. It follows by (2.1.1) that

$$V_i \subset V_\rho \otimes V_s \subset V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}} \otimes V_s \tag{2.2.11}$$

$$V_k \subset \mathbb{C} \otimes \cdots \otimes \mathbb{C} \otimes V_{\square} \tag{2.2.12}$$

$$V_j \subset V_\rho \otimes V_{\overline{s}} \subset V_{i_1} \otimes \cdots \otimes V_{i_{\ell-1}} \otimes V_{\overline{s}} \tag{2.2.13}$$

since $\mu = \rho + \sigma$ and $\lambda = \mu + \theta_j = \rho + \sigma + \theta_j$ where σ, $\sigma + \theta_j$ are weights of V_s and $V_{\overline{s}}$ respectively. Let $\psi : V_s \otimes V_{\square} \to V_{\overline{s}}$ be Clifford multiplication and $\phi = 1 \otimes \cdots \otimes 1 \otimes \psi$. We claim that if $v_\mu \in V_i \subset V_\rho \otimes V_s$ is the highest weight vector then $\phi(v_\mu \otimes f_j)$ is a non–zero highest weight vector in $V_j \subset V_\rho \otimes V_{\overline{s}}$ so that ϕ restricts to a non–zero intertwiner $V_i \otimes V_k \to V_j$. To prove our claim, assume that $\phi(v_\mu \otimes f_j) = 0$. Let $v_\rho \in V_\rho$ be the highest weight vector and $v_\sigma = \wedge_{i=1\ldots j-1} f_i \in V_s$ a vector of weight σ so that $\psi(f_{-j}) v_\sigma = 0$. Then,

$$\begin{aligned} 0 &= (\phi(v_\mu \otimes f_j), v_\rho \otimes \psi(f_j) v_\sigma) \\ &= (1 \otimes \psi(f_j) v_\mu, v_\rho \otimes \psi(f_j) v_\sigma) \\ &= (v_\mu, v_\rho \otimes \psi(f_{-j}) \psi(f_j) v_\sigma) \\ &= 2(v_\mu, v_\rho \otimes v_\sigma) \end{aligned} \tag{2.2.14}$$

in contradiction with lemma 2.1.1. Let now $e_\alpha \in \mathfrak{so}_{2n,\mathbb{C}}$ be a root vector corresponding to the simple root α. Then, $e_\alpha \phi(v_\mu \otimes f_j) = \phi(v_\mu \otimes e_\alpha f_j)$. Since $e_\alpha f_j$ has weight $\alpha + \theta_j$, this does not vanish only if $\alpha = \theta_{j-1} - \theta_j$ and in that case is proportional to $\phi(v_\mu \otimes f_{j-1})$. To conclude, it is therefore sufficient to prove that $\phi(v_\mu \otimes f_{j-1}) = 0$. To this end, notice that the weight spaces of V_s are one–dimensional and v_μ may therefore be written as

$$v_\mu = \sum v_{\rho - (\sigma' - \sigma)} \otimes v_{\sigma'} \tag{2.2.15}$$

where σ' ranges over all weights of V_s differing from σ by a sum of positive roots, $v_{\sigma'} = \wedge_{i:\sigma'_i = \frac{1}{2}} f_i$ and the $v_{\rho-(\sigma'-\sigma)} \in V_\rho$ have weight $\rho - (\sigma' - \sigma)$. Since such σ' are of the form $\sigma' = \frac{1}{2}(\theta_1 + \cdots + \theta_{j-1} - \epsilon_j \theta_j - \cdots - \epsilon_n \theta_n)$ where ϵ ranges over all even sign changes of the variables $\theta_j, \ldots, \theta_n$, we find that $f_{j-1} \wedge v_{\sigma'} = 0$ and therefore

$$\phi(v_\mu \otimes f_{j-1}) = 1 \otimes \psi(f_{j-1}) v_\mu = 0 \tag{2.2.16}$$

$\diamond$

3. Continuity of higher level primary fields

Theorem 3.1. *Let $\mathcal{H}_i, \mathcal{H}_j$ be irreducible positive energy representations at level ℓ with lowest energy subspaces V_i, V_j and V_k an irreducible* Spin_{2n}*–module admissible at level ℓ. If one of V_i, V_j, V_k is minimal, the (projectively unique) primary field $\psi : \mathcal{H}_i^{\mathrm{fin}} \otimes V_k[z, z^{-1}] \to \mathcal{H}_j^{\mathrm{fin}}$ extends to a jointly continuous operator–valued distribution $\mathcal{H}_i^\infty \otimes C^\infty(S^1, V_k) \to \mathcal{H}_j^\infty$. If, in addition, $V_k \cong \mathbb{C}^{2n}$ is the vector representation, ψ extends to a bounded map $L^2(S^1, V_k) \to \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$.*

Proof. By corollary I.2.2.3, $D = \dim(\mathrm{Hom}_{\mathrm{Spin}_{2n}}(V_i \otimes V_k, V_j)) \leq 1$. If $D = 0$, then $\psi = 0$ and there is nothing to prove. If, on the other hand $D = 1$, then by lemma 2.2.1, we have

$$V_i \subset \bigotimes_{p=1}^{\ell} V_{i_p} \qquad V_j \subset \bigotimes_{p=1}^{\ell} V_{j_p} \qquad V_k \subset \bigotimes_{p=1}^{\ell} V_{k_p} \tag{3.1}$$

where the $V_{i_p}, V_{j_p}, V_{k_p}$ are minimal and therefore admissible at level 1. Moreover, there exist intertwiners $\varphi_p \in \mathrm{Hom}_{\mathrm{Spin}_{2n}}(V_{i_p} \otimes V_{k_p}, V_{j_p})$ such that $\otimes_p \varphi_p$ restricts to a non–zero intertwiner $\varphi : V_i \otimes V_k \to V_j$. Let $\mathcal{H}_{i_p}, \mathcal{H}_{j_p}$ be the irreducible level 1 positive energy representations with lowest energy subspaces

V_{i_p} and V_{j_p} respectively. The lowest energy suspaces of the level ℓ positive energy representations

$$\mathcal{H}_{i_1,\dots,i_\ell} = \bigotimes_{p=1}^{\ell} \mathcal{H}_{i_p} \qquad\qquad \mathcal{H}_{j_1,\dots,j_\ell} = \bigotimes_{p=1}^{\ell} \mathcal{H}_{j_p} \tag{3.2}$$

contain V_i and V_j respectively and therefore, by lemma I.2.3.2, $\mathcal{H}_{i_1,\dots,i_\ell}$ and $\mathcal{H}_{j_1,\dots,j_\ell}$ contain as submodules $\mathcal{H}_i$ and $\mathcal{H}_j$ respectively. Denote by P_i and P_j the corresponding orthogonal projections. Let $\phi_p(z) : \mathcal{H}_{i_p} \otimes V_{k_p}[z, z^{-1}] \to \mathcal{H}_{j_p}$ be the level 1 primary field with initial term φ_p. Then

$$\phi(z) = P_j \phi_1(z) \otimes \cdots \otimes \phi_\ell(z) P_i \tag{3.3}$$

is easily seen to be a primary field of type $\mathcal{H}_i \otimes V_k[z, z^{-1}] \to \mathcal{H}_j$ when $V_k[z, z^{-1}]$ is embedded in $V_{k_1}[z, z^{-1}] \otimes \cdots \otimes V_{k_\ell}[z, z^{-1}]$ in the obvious way. Since the initial term of ϕ is φ, $\phi \neq 0$ and therefore, up to a non–zero multiplicative constant, $\psi = \phi$. We proceed now as in the proof of theorem 1.4.1. By theorem V.5.3.2, each ϕ_p is a product of vertex operators with uniformly bounded modes and therefore so is ϕ. It follows, by proposition 1.3.1 that ϕ extends to a jointly continuous bilinear map $C^\infty(S^1, V_k) \otimes \mathcal{H}_i^\infty \to \mathcal{H}_j^\infty$. Finally, if V_k is the vector representation we may, by lemma 2.2.1, choose $\phi(z)$ of the form

$$P_j 1 \otimes \cdots \otimes 1 \otimes \Psi(z) P_i \tag{3.4}$$

where Ψ is a level 1 vector primary field and is therefore continuous for the L^2–norm by proposition III.4.1 $\diamond$

Remark. Let ϕ, ϕ^* be a primary field and its adjoint with corresponding vertices $\begin{pmatrix} V_k \\ V_j \; V_i \end{pmatrix}$ and $\begin{pmatrix} V_k^* \\ V_i \; V_j \end{pmatrix}$. If one of V_i, V_j, V_k is minimal then, by theorem 3.1, the defining identity for ϕ^*, namely

$$(\phi(f)\xi, \eta) = (\xi, \phi^*(\overline{f})\eta) \tag{3.5}$$

where $\xi \in \mathcal{H}_i^{\text{fin}}$, $\eta \in \mathcal{H}_j^{\text{fin}}$ and $f \in V_k[z, z^{-1}]$, extends to $\xi \in \mathcal{H}_i^\infty, \eta \in \mathcal{H}_j^\infty$ and $f \in C^\infty(S^1, V_k)$. In particular,

$$\phi^*(\overline{f}) \subseteq \phi(f)^* \qquad \text{and} \qquad \phi(f) \subseteq \phi^*(\overline{f})^* \tag{3.6}$$

so that, for any $f \in C^\infty(S^1, V_k)$, $\phi(f)$ and $\phi^*(\overline{f})$ are densely defined, closeable operators.

4. Intertwining properties of smeared primary fields

Proposition 4.1. *Let $\phi : \mathcal{H}_i^{\text{fin}} \otimes V_k[z, z^{-1}] \to \mathcal{H}_j^{\text{fin}}$ be a primary field extending to an operator–valued distribution. Then, the following holds projectively on $\mathcal{H}_i^\infty$, for any $\gamma \in LG \rtimes \operatorname{Rot} S^1$ and $f \in C^\infty(S^1, V_k)$.*

$$\pi_j(\gamma)\phi(f)\pi_i(\gamma)^* = \phi(\gamma f) \tag{4.1}$$

Moreover, if d is the integrally–moded infinitesimal generator of rotations, then

$$e^{i\theta d}\phi(f)e^{-i\theta d} = \phi(f_\theta) \tag{4.2}$$

where $f_\theta(\phi) = f(\phi - \theta)$.

Proof. We use a standard ODE argument. Notice first that by continuity, ϕ intertwines the action of $L\mathfrak{g}$ on $\mathcal{H}_i^\infty$ and $\mathcal{H}_j^\infty$, *i.e.*

$$\pi_i(X)\phi(f)\xi - \phi(f)\pi_j(X)\xi = \phi(Xf)\xi \tag{4.3}$$

for any $X \in L\mathfrak{g}$, $f \in C^\infty(S^1, V_k)$, and $\xi \in \mathcal{H}_i^\infty$. Define now $F(t) = e^{-t\pi_j(X)}\phi(e^{tX}f)e^{t\pi_i(X)}\xi$ where X, f, ξ are as above. Then, using the invariance of $\mathcal{H}_i^\infty$ under LG (proposition II.1.5.3) and the

C^∞–continuity of ϕ,

$$\begin{aligned}
F(t+s) &= e^{-(t+s)\pi_j(X)}\phi(e^{tX}f + sXe^{tX}f + o(s))\Big(e^{t\pi_i(X)}\xi + s\pi_i(X)e^{t\pi_i(X)}\xi + o(s)\Big)\\
&= e^{-(t+s)\pi_j(X)}\Big(\phi(e^{tX}f)e^{t\pi_i(X)}\xi + s\phi(Xe^{tX}f)e^{t\pi_i(X)}\xi + s\phi(e^{tX}f)\pi_i(X)e^{t\pi_i(X)}\xi + o(s)\Big)\\
&= F(t) + s\Big(-\pi_j(X)\phi(e^{tX}f)e^{t\pi_i(X)}\xi + \phi(Xe^{tX}f)e^{t\pi_i(X)}\xi + s\phi(e^{tX}f)\pi_i(X)e^{t\pi_i(X)}\xi\Big) + o(s)\\
&= F(t) + o(s)
\end{aligned} \tag{4.4}$$

where $o(s)s^{-1} \to 0$ as $s \to 0$ in $\mathcal{H}^\infty$ and therefore in $\mathcal{H}$. Thus, $F \in C^1(\mathbb{R}, \mathcal{H})$ and $\dot{F} \equiv 0$ so that $F \equiv F(0) = \xi$. It follows that ϕ intertwines the one–parameter subgroups of LG and therefore LG itself. The commutation properties with $\operatorname{Rot} S^1$ follow in a similar way $\diamond$

Remark.

(i) Recall that the central extensions of $L\,\mathrm{Spin}_{2n}$ corresponding to π_i, π_j are canonically isomorphic by proposition II.2.4.3 and denote either of them by $\mathcal{L}\,\mathrm{Spin}_{2n}$. It is not difficult to show that a more astringent relations than (4.1) holds, namely the (non–projective) identity

$$\pi_j(\widetilde{\gamma})\phi(f)\phi_i(\widetilde{\gamma})^* = \phi(\widetilde{\gamma}f) \tag{4.5}$$

for any $\widetilde{\gamma} \in \mathcal{L}\,\mathrm{Spin}_{2n}$.

(ii) Somewhat conversely, Wassermann has pointed out that it might be possible to use the continuity of the level 1 spin primary fields and (4.1) to give an alternative proof of proposition II.2.4.3 for $G = \mathrm{Spin}_{2n}$. Notice first that it is only necessary to prove this at level 1 since the level ℓ representations are contained in the ℓ–fold tensor product of the level 1 representations. Moreover, we already know that the central extensions corresponding to the level 1 vacuum and vector representations are isomorphic since these arise as summands of the Neveu–Scharz Fock space (proposition III.2.5.3). The same holds for the level 1 spin representations since they are summands of the Ramond sector. Thus, it is sufficient to prove the isomorphism of the central extensions corresponding to the the vacuum representation $(\pi_0, \mathcal{H}_0)$ and one of the spin representations $(\pi_s, \mathcal{H}_s)$ at level 1. Let $\phi_s : \mathcal{H}_0^{\mathrm{fin}} \otimes V_s[z, z^{-1}] \to \mathcal{H}_s^{\mathrm{fin}}$ be the corresponding primary field. By theorem 1.4.1 and proposition 4.1, the following holds projectively for any $f \in C^\infty(S^1, V_s)$ and $\gamma \in LG$

$$\pi_s(\gamma)\phi_s(f)\pi_0(\gamma)^* = \phi_s(\gamma f) \tag{4.6}$$

This identity may be used to define a continuous map between the two central extensions by associating to any lift $\widetilde{\pi}_0(\gamma) \in U(\mathcal{H}_0)$ of $\pi_0(\gamma)$ the unique unitary lift $\widetilde{\pi}_s(\gamma)$ of $\pi_s(\gamma)$ such that (4.6) holds without any additional phase corrections. However, the proof that it is a homomorphism requires, to the best of our knowledge, a proof of the fact that the definition of the map is independent of the particular f chosen.

Part 2

Braiding of primary fields

CHAPTER VII

Braiding properties of primary fields

We outline the theory of the Knizhnik–Zamolodchikov equations satisfied by the four–point functions of primary fields and deduce from it their braiding properties. We also compute the simplest instances of braiding. Finally, we show that, when smeared on test functions, the primary fields satisfy the same braiding relations provided the supports of the functions are disjoint.

1. The Knizhnik–Zamolodchikov equations

Fix a level $\ell \in \mathbb{N}$ and consider a collection of non–zero primary fields $\phi_n \cdots \phi_1$ corresponding to the vertices

$$\begin{pmatrix} k_n \\ 0\ i_n \end{pmatrix}\begin{pmatrix} k_{n-1} \\ i_n\ i_{n-1} \end{pmatrix}\cdots\begin{pmatrix} k_2 \\ i_2\ i_1 \end{pmatrix}\begin{pmatrix} k_1 \\ i_1\ 0 \end{pmatrix} \tag{1.1}$$

Notice that $V_{k_1} = V_{i_1}$ and $V_{k_n} = V_{i_n}^*$ and that, up to a scalar the initial terms of ϕ_n and ϕ_1 are the canonical intertwiners $V_{i_n} \otimes V_{i_n}^* \to \mathbb{C}$ and $\mathbb{C} \otimes V_{i_1} \to V_{i_1}$. Fix $\Upsilon \in \mathcal{H}_0(0) \cong \mathbb{C}$ of norm one and define the *n–point function*

$$F = (\phi_n(v_n, z_n) \cdots \phi_1(v_1, z_1)\Upsilon, \Upsilon) \tag{1.2}$$

a formal Laurent series in the variables $z_n \cdots z_1$. The G-equivariance of the primary fields and the fact that G fixes Υ imply that F takes values in $(V_{k_n}^* \otimes \cdots \otimes V_{k_1}^*)^G$. We shall prove that F satisfies a first order partial differential equation. Attach for this purpose an endomorphism Ω_{ij} to any pair $1 \leq i, j \leq n$ by setting

$$\Omega_{ij} = \pi_i(X_k)\pi_j(X^k) \tag{1.3}$$

where π_i is the action of $\mathfrak{g}_{\mathbb{C}}$ on the ith factor of $(V_{k_n}^* \otimes \cdots \otimes V_{k_1}^*)^G$, $\{X_k\}, \{X^k\}$ are dual basis of $\mathfrak{g}_{\mathbb{C}}$ with respect to the basic inner product on $\mathfrak{g}_{\mathbb{C}}$ and the summation over k is implicit. Notice that Ω_{ij} is independent of the choice of the basis X_k and, in particular $\Omega_{ij} = \pi_i(X^k)\pi_j(X_k) = \Omega_{ji}$. Moreover, Ω_{ii} acts as multiplication by C_{k_i}, the Casimir of V_{k_i} and $V_{k_i}^*$.

Recall that on each $\mathcal{H}_{i_k}^{\mathrm{fin}}$, we have $L_0 = d + \Delta_{i_k}$. Here, as customary d is the normalised generator of rotations, L_0 is given by the Segal–Sugawara formula and $\Delta_{i_k} = \frac{C_{i_k}}{2\kappa}$ where C_{i_k} is the Casimir of V_{i_k} and $\kappa = \ell + \frac{C_{\mathfrak{g}}}{2}$. Therefore, by (I.4.4),

$$z_i \frac{d}{dz_i}\phi_i(v_i, z_i) = [d, \phi_i(v_i, z_i)] - \Delta_{\phi_i}\phi_i(v_i, z_i) = [L_0, \phi_i(v_i, z_i)] - \Delta_{k_i}\phi_i(v_i, z_i) \tag{1.4}$$

Using the explicit expression of L_0,

$$L_0 = \frac{1}{\kappa}\Big(\frac{1}{2}X_i(0)X^i(0) + \sum_{m>0} X_i(-m)X^i(m)\Big) \tag{1.5}$$

and the equivariance properties of primary fields, we find

$$\begin{aligned}\kappa(z_i\frac{d}{dz_i} + \Delta_{k_i})F =&\frac{1}{2}(\prod_{j>i}\phi_j(v_j, z_j)[X_k(0)X^k(0), \phi_i(v_i, z_i)]\prod_{i>j}\phi_j(v_j, z_j)\Upsilon, \Upsilon)\\ &+ \sum_{m>0}(\prod_{j>i}\phi_j(v_j, z_j)[X_k(-m)X^k(m), \phi_i(v_i, z_i)]\prod_{i>j}\phi_j(v_j, z_j)\Upsilon, \Upsilon)\\ =&\frac{1}{2}C_{k_i}F + \sum_{i>j}\Omega_{ij}F + \sum_{m>0}\left(\sum_{j>i} - \left(\frac{z_i}{z_j}\right)^m\Omega_{ij}F + \sum_{i>j}\left(\frac{z_j}{z_i}\right)^m\Omega_{ij}F\right)\end{aligned} \tag{1.6}$$

where the last equality is obtained by moving the $X_k(p)$, $p<0$ to the left and the $X^k(q)$, $q\geq 0$ to the right and using $X(p)\Upsilon=0$ if $p\geq 0$. Thus, with the understanding that the denominators below are to be expanded as if $|z_n|>\cdots>|z_1|$, F satisfies the *Knizhnik–Zamolodchikov equations*

$$\frac{\partial F}{\partial z_i}=\frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}F \tag{1.7}$$

Notice that $\Delta_{\phi_1}=0$ so that $\phi_1(v_1,z_1)\Upsilon=\sum_{n\geq 0}z^{-n}\phi_1(v_1,n)\Upsilon$ and, formally $\lim_{z_1\to 0}\phi_1(v_1,z_1)\Upsilon=v_1$. Similarly, $\lim_{z_n\to\infty}(z_n^{\Delta_{\phi_n}}\phi_n(v_n,z_n)\cdots,\Upsilon)=(\cdots,\overline{v_n})$ where we have used the antilinear identification $V_{k_n}^*\to V_{k_n}$. Thus, the function

$$H(z_2,\dots z_{n-1})=(\phi_{n-1}(v_{n-1},z_{n-1})\cdots\phi_2(v_2,z_2)v_1,\overline{v_n})=\lim_{\substack{z_1\to 0\\ z_n\to\infty}}z_n^{\Delta_{\phi_n}}F \tag{1.8}$$

satisfies, for $i=2\dots n-1$ the equations

$$\frac{\partial H}{\partial z_i}=\frac{1}{\kappa}\Big(\frac{\Omega_{i1}}{z_i}+\sum_{j\neq 1,i,n}\frac{\Omega_{ij}}{z_i-z_j}\Big)H \tag{1.9}$$

We specialise now to the case $n=4$. Set $w=z_3$ and $z=z_2$, then $(w\partial_w+z\partial_z)H=\kappa^{-1}(\Omega_{12}+\Omega_{23}+\Omega_{31})H$. However, on $(V_{k_1}^*\otimes\cdots\otimes V_{k_4}^*)^G$, we have $\sum_{i=1}^4\pi_i(X)=0$ for any $X\in\mathfrak{g}$ and therefore

$$\begin{aligned}\Omega_{12}+\Omega_{23}+\Omega_{31}&=\frac{1}{2}(\sum_{i=1}^3\pi_i(X_k))(\sum_{i=1}^3\pi_i(X^k))-\frac{1}{2}(\Omega_{11}+\Omega_{22}+\Omega_{33})=\frac{1}{2}(\Omega_{44}-\Omega_{11}-\Omega_{22}-\Omega_{33})\\&=\kappa(\Delta_{k_4}-\Delta_{k_1}-\Delta_{k_2}-\Delta_{k_3})=-\kappa(\Delta_{\phi_2}+\Delta_{\phi_3})\end{aligned} \tag{1.10}$$

It follows that the *reduced four point function* $\Psi(w,z)=w^{\Delta_{\phi_3}}z^{\Delta_{\phi_2}}H$ satisfies $w\partial_w\Psi=-z\partial_z\Psi$ and therefore only contains monomials in $\zeta=zw^{-1}$. Moreover, since $\zeta d_\zeta\Psi=z\partial_z\Psi$, Ψ satisfies as a function of one formal variable

$$\frac{d\Psi}{d\zeta}=\frac{1}{\kappa}\Big(\frac{\Omega_{12}+\kappa\Delta_{\phi_2}}{\zeta}+\frac{\Omega_{23}}{\zeta-1}\Big)\Psi \tag{1.11}$$

where $(\zeta-1)^{-1}$ is to be expanded as if $|\zeta|<1$.

Aside from (1.11), Ψ satisfies a number of algebraic identities. To see this, let $v_1\in V_{k_1}$ be a highest weight vector for $\{e_\theta,f_\theta,h_\theta\}$ with $h_\theta v_1=s_1v_1$. Since v_1 is of lowest energy, $f_\theta(1)v_1=0$ and therefore v_1 is of highest weight for $\{f_\theta(1),e_\theta(-1),\ell-h_\theta\}$ with eigenvalue $\ell-s_1$. Thus, by elementary $\mathfrak{sl}_2(\mathbb{C})$ theory, $e_\theta(-1)^{\ell-s_1+1}v_1=0$. On the other hand, a simple induction shows that, for any $n\in\mathbb{N}$

$$(\phi_3(v_3,w)\phi_2(v_2,z)e_\theta(-1)^nv_1,v_4)=(-1)^n(w^{-1}\pi_3(e_\theta)+z^{-1}\pi_2(e_\theta))^n(\phi_3(v_3,w)\phi_2(v_2,z)v_1,v_4) \tag{1.12}$$

so that the right hand–side vanishes for $n=\ell-s_1+1$. An additional set of identities for Ψ may be similarly derived from the equation $e_\theta(-1)^{\ell-s_4+1}v_4=0$ which holds whenever $v_4\in V_4$ is of highest weight for $\{e_\theta,f_\theta,h_\theta\}$ with $h_\theta v_4=s_4v_4$. This completes the proof of the first part of the following

Proposition 1.1 (Tsuchiya-Kanie). *Let ϕ_3,ϕ_2 be primary fields with vertices $\begin{pmatrix}k_3\\k_4\ j\end{pmatrix}$, $\begin{pmatrix}k_2\\j\ k_1\end{pmatrix}$ and conformal weights Δ_3, Δ_2 respectively. Define the reduced four–point function*

$$\Psi_j=w^{\Delta_3}z^{\Delta_2}(\phi_3(v_3,w)\phi_2(v_2,z)v_1,v_4)=\sum_{n\geq 0}(\psi_3(v_3,n)\phi_2(v_2,-n)v_1,v_4)\left(\frac{z}{w}\right)^n \tag{1.13}$$

a formal power series in the variable $\zeta=\left(\frac{z}{w}\right)$ with coefficents in $(V_{k_4}\otimes V_{k_3}^*\otimes V_{k_2}^*\otimes V_{k_1}^*)^G$. Then Ψ_j converges to a holomorphic function on $|\zeta|<1$ such that $\Phi_j=\zeta^{-\Delta_2}\Psi_j$ satisfies

$$\frac{d\Phi_j}{d\zeta}=\frac{1}{\kappa}\Big(\frac{\Omega_{12}}{\zeta}+\frac{\Omega_{23}}{\zeta-1}\Big)\Phi_j \tag{1.14}$$

and the algebraic equations

$$(\pi_2(e_\theta)+\zeta\pi_3(e_\theta))^{\ell-s_1+1}\Phi_j(v_1,v_2,v_3,v_4)=0 \tag{1.15}$$

whenever v_1 is a highest weight vector for $\{e_\theta,f_\theta,h_\theta\}$ with $h_\theta v_1=s_1v_1$, and

$$(\zeta\pi_2(f_\theta)+\pi_3(f_\theta))^{\ell-s_4+1}\Phi_j(v_1,v_2,v_3,v_4)=0 \tag{1.16}$$

whenever v_4 is a highest weight vector for $\{e_\theta,f_\theta,h_\theta\}$ with $h_\theta v_1=s_4v_1$. The set of Φ_j corresponding to all intermediate V_j is a basis of the space of solutions to the above system of equations near 0.

Remark. The convergence of Ψ_j follows at once because Ψ_j is a formal power series solution of (1.14), which has a regular singularity at $\zeta=0$. The completeness of the set of reduced four–point functions is stated as Proposition 4.3 of [**TK1**]. Its proof contains a gap bridged by the treatment given in [**Wa2**].

2. Braiding of primary fields

Theorem 2.1 (Braiding). *Let $\phi^{k_3}_{k_4 j},\phi^{k_2}_{jk_1}$ be primary fields corresponding to the vertices $\begin{pmatrix} k_3 \\ k_4\ j\end{pmatrix}$ and $\begin{pmatrix} k_2 \\ j\ k_1\end{pmatrix}$. Then the product $\phi^{k_3}_{k_4 j}(v_3,w)\phi^{k_2}_{jk_1}(v_2,z)$ defines a single–valued holomorphic function on $\{(w,z)|\frac{w}{z}\notin[0,\infty)\}$ when it is evaluated between finite energy vectors. Moreover,*

$$\phi^{k_3}_{k_4 j}(v_3,w)\phi^{k_2}_{jk_1}(v_2,z)=\sum_h b_h\phi^{k_2}_{k_4 h}(v_2,z)\phi^{k_3}_{hk_1}(v_3,w) \tag{2.1}$$

where $b_h\in\mathbb{C}$ and the sum on the right hand–side spans the products of all primary fields with vertices $\begin{pmatrix} k_2 \\ k_4\ h\end{pmatrix}$, $\begin{pmatrix} k_3 \\ h\ k_1\end{pmatrix}$ corresponding to any intermediate V_h.

Proof. By the commutation properties of the primary fields with $L^{\mathrm{pol}}\mathfrak{g}$, it is sufficient to prove (2.1) when it is evaluated between lowest energy vectors. Let Ψ_j be the reduced four point-function in the variable $\zeta=\frac{z}{w}$ corresponding to $\phi^{k_3}_{k_4 j}\phi^{k_2}_{jk_1}$. By the previous proposition, $\Phi_j=\zeta^{-\Delta_2}\Psi_j$, $\Delta_2=\Delta_{k_1}+\Delta_{k_2}-\Delta_j$ converges to a homolorphic function on $|\zeta|<1$ which may analytically be continued to a single–valued function on $\mathbb{C}\backslash[0,\infty]$ satisfying (1.14)–(1.16). Near ∞, we may rewrite (1.14) in terms of the local coordinate $\eta=\zeta^{-1}$. Since $\zeta\partial_\zeta=-\eta\partial_\eta$, Φ_j satisfies

$$\frac{d\Phi_j}{d\eta}=\frac{1}{\kappa}\Big(-\frac{\Omega_{12}}{\eta}+\frac{\Omega_{23}}{\eta(\eta-1)}\Big)\Phi_j=\frac{1}{\kappa}\Big(\frac{\Omega_{13}+\kappa(\Delta_2+\Delta_3)}{\eta}+\frac{\Omega_{23}}{\eta-1}\Big)\Phi_j \tag{2.2}$$

where we used (1.10). On the other hand, the boundary conditions may be rewritten in terms of η as

$$(\eta\pi_2(e_\theta)+\pi_3(e_\theta))^{\ell-s_1+1}\Phi_j(v_1,v_2,v_3,v_4)=0 \tag{2.3}$$

$$(\pi_2(f_\theta)+\eta\pi_3(f_\theta))^{\ell-s_4+1}\Phi_j(v_1,v_2,v_3,v_4)=0 \tag{2.4}$$

It follows that $\Phi_j\eta^{-(\Delta_2+\Delta_3)}=\Psi_j\zeta^3$ is a solution of (1.14)–(1.16) with respect η when $(V_{k_4}\otimes V^*_{k_3}\otimes V^*_{k_2}\otimes V^*_{k_1})^G$ and $(V_{k_4}\otimes V^*_{k_2}\otimes V^*_{k_3}\otimes V^*_{k_1})^G$ are identified in the obvious way. By proposition 1.1, it may be therefore be written as a sum of reduced four point functions corresponding to the vertices $\begin{pmatrix} k_2 \\ k_4\ h\end{pmatrix}$, $\begin{pmatrix} k_3 \\ h\ k_1\end{pmatrix}$ multiplied by an appropriate power of η. Writing $\zeta=zw^{-1}$ and $\eta=wz^{-1}$, the powers of z and w are easily seen to cancel and (2.1) is proved $\diamond$

Remark.

(i) The braiding coefficients b_h in (2.1) are only defined up to multiplication by non–zero constants accounting for the multiplicative ambiguity in the choice of the initial term of the various primary fields. A further lack of determination arises if the Hom spaces corresponding to some of the vertices are of dimension greater or equal to 2. However, once a definite choice of initial terms is made, the b_h are unambiguously defined and may be computed by expressing the analytic continuation of the reduced four–point function corresponding to

the left hand–side of (2.1) from $\zeta = 0$ to ∞ in terms of the reduced four point functions corresponding to the right hand–side evaluated at ζ^{-1}.

(ii) The existence of braiding is perhaps best demonstrated using the Knizhnik–Zamolodchikov equations in their PDE form. These define a flat connection on the configuration space of n ordered points in $\mathbb{C}$ for the (topologically) trivial bundle with fibre $(V_{k_n}^* \otimes \cdots \otimes V_{k_1}^*)^G$. The permutation of the fields ϕ_i and ϕ_{i+1} is then obtained by analytic continuation along a path connecting $(z_n \ldots z_{i+1}, z_i \ldots z_1)$ to $(z_n \ldots z_i, z_{i+1} \ldots z_1)$. The above one–variable approach is possible because the connection is invariant under the diagonal action of $SL(2, \mathbb{C})$ and gives an effective method for computing the braiding coefficients.

3. Abelian braiding

The simplest instance of braiding is the *abelian braiding* obtained from Knizhnik–Zamolodchikov equations with values in a one–dimensional space. We compute it below

LEMMA 3.1. *Let V_i, $i = 2 \ldots 4$ be irreducible G–modules such that $V_4 \subset V_2 \otimes V_3$ with multiplicity one and assume that the generator of* $\mathrm{Hom}_G(V_2 \otimes V_3, V_4)$ *is the initial term of a primary field $\phi_{V_4\,V_2}^{V_3}$. Then,*

$$\phi_{V_4\,V_2}^{V_3}(w)\phi_{V_2\,V_0}^{V_2}(z) = \lambda \phi_{V_4\,V_3}^{V_2}(z)\phi_{V_3\,V_0}^{V_3}(w) \tag{3.1}$$

where $\lambda \neq 0$. In fact, let the initial terms of $\phi_{V_i\,V_0}^{V_i}$, $i = 2, 3$ be the canonical intertwiners $\mathbb{C} \otimes V_i \to V_i$ and denote those of $\phi_{V_4\,V_2}^{V_3}$ and $\phi_{V_4\,V_3}^{V_2}$ by φ_3 and φ_2 respectively. Then

(i) *If φ_3, φ_2 are normalised by setting $\varphi_2 = \varphi_3 \sigma$ where $\sigma : V_3 \otimes V_2 \to V_2 \otimes V_3$ is permutation, then $\lambda = e^{-i\pi\beta}$ where $\beta = \frac{1}{2\kappa}(C_4 - C_2 - C_3)$ and the C_i are the Casimirs of the corresponding representations.*

(ii) *If $V_3 = V_2$, and φ_3, φ_2 are normalised by equating them so that $\phi_{4\,2}^3 = \phi_{4\,3}^2$, then $\lambda = \varepsilon e^{-i\pi\beta}$ where $\varepsilon = \pm 1$ according to whether $V_4 \subset V_2 \otimes V_2$ is symmetric or anti-symmetric under the action of $\mathfrak{S}_2$.*

PROOF. By assumption, $W = (V_4 \otimes V_3^* \otimes V_2^* \otimes \mathbb{C})^G \cong \mathrm{Hom}_G(V_2 \otimes V_3, V_4)$ is one–dimensional and the Knizhnik–Zamolodchikov equations with values in W have a unique non trivial solution f given by the reduced four–point function of the left–hand side of (3.1). The vanishing of λ would lead to that of f, a contradiction.

(i) Ω_{23} acts on W as multiplication by $\kappa\beta$. Indeed,

$$\begin{aligned}
\Omega_{23}P &= P\pi_2(X_k)\pi_3(X^k) \\
&= \frac{1}{2}P(\pi_2(X_k) + \pi_3(X_k))(\pi_2(X^k) + \pi_3(X^k)) - \frac{1}{2}P(\pi_2(X_k)\pi_2(X^k) + \pi_3(X_k)\pi_3(X^k)) \\
&= \frac{1}{2}\pi_4(X_k)\pi_4(X^k)P - \frac{1}{2}P(C_2 + C_3) \\
&= \frac{1}{2}(C_4 - C_2 - C_3)P
\end{aligned} \tag{3.2}$$

Thus, the Knizhnik–Zamolodchikov equations read $f' = \beta(\zeta-1)^{-1}f$ and, up to a constant, $f = (\zeta-1)^\beta$ where, in accordance with our conventions on the definition of braiding, the function is defined with a cut in the $\zeta - 1$ plane along $[0, \infty]$ so that f is continuous on $\mathbb{C}\backslash[0, \infty]$. Since the leading term of f at 0 is φ_3, we get $f = e^{-\pi i\beta}(z-1)^\beta\varphi_3$. Similarly, the leading coefficient at ∞ of the reduced four point-function g corresponding to the right hand–side of (3.1) is φ_2 so that $g(\eta) = e^{-\pi i\beta}(\eta - 1)^\beta\varphi_2$ where $\eta = \zeta^{-1}$. If $\zeta \in (-\infty, 0)$ so that $\arg\zeta = \pi$, then $(\zeta - 1)^\beta = e^{-i\pi\beta}\zeta^\beta(1/\zeta - 1)^\beta$ and therefore the analytic continuation of $f\sigma$ at ∞ is $e^{-2\pi i\beta}\eta^{-\beta}(\eta - 1)^\beta\varphi_3\sigma$ whence $\lambda = e^{-2\pi i\beta}$ as claimed.

(ii) follows at once from (i) since $\varphi_3\sigma = \varepsilon\varphi_3$ ◇

4. Braiding of smeared primary fields

PROPOSITION 4.1. *Let ϕ_{kj}, ϕ_{ji}, ϕ_{kh}, ϕ_{hi} be primary fields of charges U, V, V, U respectively satisfying the braiding relations*

$$\phi_{kj}(u,w)\phi_{ji}(v,z) = \sum_h b_h \phi_{kh}(v,z)\phi_{hi}(u,w) \tag{4.1}$$

If all fields extend to operator–valued distributions, then for any $f \in C^\infty(S^1, U)$ and $g \in C^\infty(S^1, V)$ supported in $I = (0,\pi)$ and $I^c = (\pi, 2\pi)$ respectively, the following holds

$$\phi_{kj}(f)\phi_{ji}(g) = \sum_h b_h \phi_{kh}(g e_{\alpha_{jh}})\phi_{hi}(f e_{-\alpha_{jh}}) \tag{4.2}$$

where $\alpha_{jh} = \Delta_i + \Delta_k - \Delta_j - \Delta_h$ and $e_\mu(\theta) = e^{i\mu\theta}$ for $\theta \in (0, 2\pi)$.

PROOF. By continuity, we need only prove (4.2) when both sides are evaluated between finite energy vectors and by the commutation relations of the primary fields with $L^{\mathrm{pol}}\mathfrak{g}$ it is sufficient to consider lowest energy vectors. Moreover, we may assume that f and g are of the form $a \otimes u$, $b \otimes v$ where $a, b \in C^\infty(S^1, \mathbb{C})$ are supported in I, I^c respectively and $(u, v) \in U \times V$. Let ξ, η be lowest energy vectors, then

$$(\phi_{kj}(f)\phi_{ji}(g)\xi, \eta) = \sum_{n,m} a_n b_m (\phi_{kj}(u,n)\phi_{ji}(v,m)\xi, \eta) = \sum_{n \geq 0} a_n b_{-n} (\phi_{kj}(u,n)\phi_{ji}(v,-n)\xi, \eta) \tag{4.3}$$

an absolutely convergent sum since the Fourier coefficents a_n, b_m of a, b decrease rapidly and, by the continuity of ϕ_{kj}, ϕ_{ji}, $\|\phi_{ji}(u,n)\xi\| \leq C(1+|n|)^{\beta_{ji}}$, $\|\phi_{kj}(v,m)\eta\| \leq C(1+|m|)^{\beta_{kj}}$ for some $\beta_{ji}, \beta_{kj} \geq 0$. Let

$$F_j(\zeta) = \sum_{n \geq 0} (\phi_{kj}(u,n)\phi_{ji}(v,-n)\xi, \eta)\zeta^n \tag{4.4}$$

be the reduced four–point function corresponding to the product $\phi_{kj}\phi_{ji}$. F_j is convergent on $|\zeta| < 1$ and extends to a single–valued holomorphic function on $\mathbb{C}\backslash[0,\infty)$. The function $\check{a}*b(\theta) = \int_0^{2\pi} a(-\phi)b(\theta-\phi)\frac{d\phi}{2\pi}$ has Fourier coefficients $a_{-n}b_n$ and is supported away from 0 so that $\check{a} * b(\zeta|\zeta|^{-1})F_j(\zeta)$ is smooth on $\mathbb{C}\backslash\{0\}$. Thus, (4.3) is equal to

$$\lim_{r \nearrow 1} \int_0^{2\pi} \check{a} * b(\theta) F_j(re^{i\theta})\frac{d\theta}{2\pi} = \int_0^{2\pi} \check{a} * b(\theta) F_j(e^{i\theta})\frac{d\theta}{2\pi} = \lim_{r \searrow 1} \int_0^{2\pi} \check{a} * b(\theta) F_j(re^{i\theta})\frac{d\theta}{2\pi} \tag{4.5}$$

Let now G_h be the reduced four–point functions corresponding to the products $\phi_{kh}\phi_{hi}$. From the braiding relations (4.1), we have $F_j(\zeta) = \sum_h b_h \zeta^{\alpha_{jh}} G_h(\zeta^{-1})$ where $\alpha_{jh} = \Delta_i + \Delta_k - \Delta_j - \Delta_h$ and (4.5) is therefore equal to

$$\begin{aligned} &\sum_h b_h \lim_{r \searrow 1} r^{\alpha_{jh}} \int_0^{2\pi} \check{a} * b\, e_{\alpha_{jh}}(\theta) G_h(r^{-1}e^{-i\theta})\frac{d\theta}{2\pi} \\ =&\sum_h b_h \lim_{r \searrow 1} r^{\alpha_{jh}} \int_0^{2\pi} e^{2\pi i \alpha_{jh}\theta} \check{b} * a\, e_{-\alpha_{jh}}(\theta) G_h(r^{-1}e^{i\theta})\frac{d\theta}{2\pi} \end{aligned} \tag{4.6}$$

Since $(\check{b} * a)e_{-\mu} = e^{-2\pi i\mu}(\check{be_\mu}) * (ae_{-\mu})$, the above yields, by the continuity of ϕ_{kh} and ϕ_{hi},

$$\sum_h b_h \sum_{n \geq 0} (be_{\alpha_{jh}})_n (ae_{-\alpha_{jh}})_{-n} (\phi_{kh}(v,n)\phi_{hi}(u,-n)\xi, \eta) = \sum_h b_h (\phi_{kh}(g e_{\alpha_{jh}})\phi_{hi}(f e_{-\alpha_{jh}})\xi, \eta) \tag{4.7}$$

as claimed $\diamond$

CHAPTER VIII

Braiding the vector primary field with its symmetric powers

In this chapter, we compute the structure constants governing the braiding of the $L\,\mathrm{Spin}_{2n}$–primary fields whose charges are the vector representation and one of its symmetric (traceless) powers. As explained in chapter VII, the braiding coefficients may be obtained via the analytic continuation from 0 to ∞ of the solutions of the underlying Knizhnik–Zamolodchikov equation. In section 1, we study this problem for a related third order Fuchsian ODE discovered by Dotsenko and Fateev. Its solutions are expressed as generalised Euler integrals and their analytic continuation computed by contour deformation. In section 2, we compute explicitly the matrices Ω_{ij} of the Knizhnik–Zamolodchikov equation. Finally, in section 3 we show that the Knizhnik–Zamolodchikov equation reduces to the Dotsenko–Fateev equation and deduce the analytic continuation results for the former from those of the latter.

1. The Dotsenko–Fateev equation

The Dotsenko–Fateev equation is the following third order Fuchsian ordinary differential equation with regular singular points at $0, 1, \infty$ [**DF**, eqn. (A.9)]

$$f''' + \frac{K_1 z + K_2(z-1)}{z(z-1)} f'' + \frac{L_1 z^2 + L_2(z-1)^2 + L_3 z(z-1)}{z^2(z-1)^2} f' + \frac{M_1 z + M_2(z-1)}{z^2(z-1)^2} f = 0 \tag{1.1}$$

where

$$K_1 = -(g + 3b + 3c) \qquad K_2 = -(g + 3a + 3c) \tag{1.2}$$

$$L_1 = (b+c)(2b + 2c + g + 1) \qquad L_2 = (a+c)(2a + 2c + g + 1) \tag{1.3}$$

$$\begin{aligned} L_3 = {} & (b+c)(2a + 2c + g + 1) + (a+c)(2b + 2c + g + 1) \\ & + (c-1)(a+b+c) + (3c+g)(a+b+c+g+1) \end{aligned} \tag{1.4}$$

$$M_1 = -c(2b + 2c + g + 1)(2a + 2b + 2c + g + 2) \quad M_2 = -c(2a + 2c + g + 1)(2a + 2b + 2c + g + 2) \tag{1.5}$$

and $a, b, c, g \in \mathbb{C}$ are free parameters. In §1.2, we prove that if a, b, c, g lie in a suitable range, the generalised Euler integrals

$$\int_{C_1}\int_{C_2} t_1^a (t_1 - 1)^b (t_1 - z)^c t_2^a (t_2 - 1)^b (t_2 - z)^c (t_1 - t_2)^g \, dt_1 dt_2 \tag{1.6}$$

where the C_i are contours joining one of the points $0, z, 1, \infty$ to another, yield solutions of (1.1). Following [**DF**], we show in §1.3 how different choices of the C_i lead to a basis of solutions diagonalising the monodromy at a given singular point and in §1.4 compute the analytic continuation of solutions from 0 to ∞ by deforming the contours. We begin by establishing the convergence of (1.6).

1.1. Convergence of contour integrals.

We shall need a special case of a more general result of Selberg [**Sel**]

Proposition 1.1.1 (Selberg). *The improper integral*

$$J_2(\alpha, \beta; \gamma) = \int_0^1 \int_0^1 t_1^\alpha (1 - t_1)^\beta t_2^\alpha (1 - t_2)^\beta |t_1 - t_2|^\gamma \, dt_1 dt_2 \tag{1.1.1}$$

is absolutely convergent if

$$\Re\alpha, \Re\beta, \Re\gamma > -1 \qquad \Re(2\alpha+\gamma) > -2 \qquad \Re(2\beta+\gamma) > -2 \tag{1.1.2}$$

and is equal to

$$\prod_{j=1}^{2} \frac{\Gamma(j\frac{\gamma}{2}+1)\Gamma(\alpha+(j-1)\frac{\gamma}{2}+1)\Gamma(\beta+(j-1)\frac{\gamma}{2}+1)}{\Gamma(\frac{\gamma}{2}+1)\Gamma(\alpha+\beta+j\frac{\gamma}{2}+2)} \tag{1.1.3}$$

Remark. If α, β, γ satisfy (1.1.2) then $J_2(\alpha, \beta; \gamma) \neq 0$ since $\Gamma(\zeta)^{-1}$ vanishes iff $\zeta \in \{0, -1, -2, \ldots\}$ and $\Re(\frac{\gamma}{2}+1) > \frac{1}{2}$, $\Re(\alpha+\beta+\frac{\gamma}{2}+2) = \Re(\alpha+\frac{1}{2}(2\beta+\gamma)+2) > 0$, $\Re(\alpha+\beta+\gamma+2) = \Re(\frac{1}{2}(2\alpha+\gamma)+\frac{1}{2}(2\beta+\gamma)+2) > 0$.

Corollary 1.1.2. *Let $a, b, c, g \in \mathbb{C}$ satisfy*

$$\Re a, \Re b, \Re c, \Re g > -1 \qquad \Re(2a+g), \Re(2b+g), \Re(2c+g) > -2 \tag{1.1.4}$$

$$\Re(a+b+c+g) < -1 \qquad \Re(2a+2b+2c+g) < -2 \tag{1.1.5}$$

Then, for any pair of contours $C_i : I \to \mathbb{P}^1$ with interior $C_i(\mathring{I}) \subset \mathbb{P}^1 \setminus \{0, 1, z, \infty\}$ and end–points $C_i(\partial I) \in \{0, 1, z, \infty\}$, the improper integral

$$\int_{C_1}\int_{C_2} t_1^a(t_1-1)^b(t_1-z)^c t_2^a(t_2-1)^b(t_2-z)^c(t_1-t_2)^g dt_1 dt_2 \tag{1.1.6}$$

is absolutely convergent, locally uniformly in $z \in \mathbb{C} \setminus \{0, 1\}$.

Proof. This follows by a tedious case–by–case analysis. When both C_1 and C_2 have their end–points at 0 and 1, the result is an immediate corollary of proposition 1.1.1 since $|(t_1-z)^g(t_2-z)^g|$ is bounded on $C_1 \times C_2$ locally uniformly in z. The other cases follow in a similar fashion after a change of variable bringing the end–points of C_1 and C_2 onto $0, 1$ ◇

1.2. A derivation of the Dotsenko–Fateev equation.

We prove in proposition 1.2.1 that the integrals (1.6) are solutions of the Dotsenko–Fateev equation. The proof proceeds as follows. We consider the multi–valued function

$$\Phi = t^a(t-1)^b(t-z)^c s^a(s-1)^b(s-z)^c(t-s)^g \tag{1.2.1}$$

and associated holomorphic differential form $\omega = \Phi dt \wedge ds$ on $C_z = \{(t, s) \in \mathbb{C}^2 | \, t, s \notin \{0, 1, z\}, t \neq s\}$ and prove that the cohomology class of ω satisfies (1.1). Integration by parts then shows that $\int_{C_1 \times C_2} \omega$ satisfies (1.1).

A word of explanation is owed to the geometrically minded reader. A more elegant approach to the above, indeed one that yields solutions of (1.1) for all values of a, b, c, g, is to consider over each fibre of $C_z \to C \to \mathbb{C} \setminus \{0, 1\} \ni z$ the flat line bundle $\mathcal{L}$ determined by Φ. ω may then be interpreted as a section of the cohomology bundle whose fibre at z is $H^2(C_z, \mathcal{L})$ and correspondingly it should be integrated over homology cycles in $H_2(C_z, \mathcal{L}^\vee)$ of which the $C_1 \times C_2$ we have chosen are not elements. Explicit basis elements of $H_2(C_z, \mathcal{L}^\vee)$ are described in [**Ko**] but leave one with the problem of showing that these may be deformed to the more singular contours we have been using. This is necessary for the computations of §1.4 below critically depend upon the use of these simpler contours. On the other hand, the deformation result requires a substantial amount of book–keeping, see for example [**TK2**, §5] where a somewhat simpler case is treated. We have therefore opted for a more bare–handed and direct approach.

The proof of proposition 1.2.1 relies on the computation of the exterior derivative d of a number of differential forms of the kind Φf where f is meromorphic. These are more easily spelled out by omitting Φ and using the twisted differential $\widetilde{d} = d + d\log\Phi$ given by

$$\widetilde{d} = d + (\frac{a}{t} + \frac{b}{t-1} + \frac{c}{t-z} + \frac{g}{t-s})dt + (\frac{a}{s} + \frac{b}{s-1} + \frac{c}{s-z} + \frac{g}{s-t})ds \tag{1.2.2}$$

since $d(\Phi f) = (\Phi d + \Phi d \log \Phi) f = \Phi \widetilde{d} f$. We begin with a digression.

The prototype of the derivation of the Dotsenko–Fateev equation is that of the hypergeometric one satisfied by the integral

$$\int_1^\infty t^a (t-1)^b (t-z)^c dt \tag{1.2.3}$$

which we briefly sketch. The basic multi-valued function is $\Phi = t^a(t-1)^b(t-z)^c$ with corresponding twisted differential $\widetilde{d} = d + (\frac{a}{t} + \frac{b}{t-1} + \frac{c}{t-z})dt$. Given our convention on omitting Φ, differentiation by z is given by $\nabla = \frac{d}{dz} - \frac{c}{t-z}$ so that the basic class $\omega = \Phi dt$ and its derivatives $\omega^{(k)} = \nabla^k \omega$ are

$$\omega = dt \qquad \omega' = -c\frac{dt}{t-z} \qquad \omega'' = c(c-1)\frac{dt}{(t-z)^2} \tag{1.2.4}$$

The hypergeometric equation may be obtained almost at one stroke by writing

$$\begin{aligned} 0 = \widetilde{d}\Big(\frac{1}{t-z}\Big) &= -\frac{dt}{(t-z)^2} + (\frac{a}{t} + \frac{b}{t-1} + \frac{c}{t-z})\frac{dt}{t-z} \\ &= c^{-1}\omega'' - (\frac{a}{z} + \frac{b}{z-1})c^{-1}\omega' - \frac{a}{z}\frac{dt}{t} - \frac{b}{z-1}\frac{dt}{t-1} \end{aligned} \tag{1.2.5}$$

The last two terms can be eliminated from (1.2.5) by using

$$0 = \widetilde{d}(t-1) = dt + (a\frac{t-1}{t} + b + c\frac{t-1}{t-z})dt = (1+a+b+c)\omega - (z-1)\omega' - a\frac{dt}{t} \tag{1.2.6}$$

and

$$0 = \widetilde{d}t = dt + (a + b\frac{t}{t-1} + c\frac{t}{t-z})dt = (1+a+b+c)\omega - z\omega' + b\frac{dt}{t-1} \tag{1.2.7}$$

Substituting (1.2.6)–(1.2.7) into (1.2.5) yields

$$\omega'' - \frac{(b+c)z + (a+c)(z-1)}{z(z-1)}\omega' + \frac{c(1+a+b+c)}{z(z-1)}\omega = 0 \tag{1.2.8}$$

so that the cohomology class of ω satisfies the hypergeometric equation

$$\omega'' + \frac{(1+\alpha+\beta)z - \gamma}{z(z-1)}\omega' + \alpha\beta\omega = 0 \tag{1.2.9}$$

with $\alpha = -c$, $\beta = -(1+a+b+c)$ and $\gamma = -(a+c)$ [**In**, §7.23].

We turn now to the differential equation satisfied by the contour integrals

$$\int_{C_1}\int_{C_2} t^a(t-1)^b(t-z)^c s^a(s-1)^b(s-z)^c(t-s)^g dtds \tag{1.2.10}$$

The twisted differential is given by (1.2.2) and differentiation by z acts as $\nabla = \frac{d}{dz} - c(\frac{1}{t-z} + \frac{1}{s-z})$. Denoting the anti-symmetrisation of a differential form ϕ with respect to the permutation $\sigma(t,s) = (s,t)$ by $[\phi] = \phi - \sigma^*\phi$, the basic class and its derivatives are

$$\omega = dt\wedge ds \qquad \omega' = -c[\frac{dt\wedge ds}{t-z}] \qquad \omega'' = c(c-1)[\frac{dt\wedge ds}{(t-z)^2}] + c^2[\frac{dt\wedge ds}{(t-z)(s-z)}] \tag{1.2.11}$$

and

$$\omega''' = -c(c-1)(c-2)[\frac{dt\wedge ds}{(t-z)^3}] - 3c^2(c-1)[\frac{dt\wedge ds}{(t-z)^2(s-z)}] \tag{1.2.12}$$

Proposition 1.2.1. *Let $a, b, c, g \in \mathbb{C}$ lie in the range* (1.1.4)–(1.1.5) *so that the integrals*

$$\int_{C_1}\int_{C_2} t^a(t-1)^b(t-z)^c s^a(s-1)^b(s-z)^c(t-s)^g dtds \tag{1.2.13}$$

where C_1 and C_2 are as in corollary 1.1.2, converge and define multi–valued holomorphic functions of z. Then, these satisfy the Dotsenko–Fateev equation (1.1).

PROOF. The strategy is similar to that used for the hypergeometric equation above. We start from a given cohomological identity involving ω and some additional terms and gradually eliminate these by using other cohomological identities. Write

$$\begin{aligned}
0=\widetilde{d}[\frac{ds}{(t-z)^2}] &= (c-2)[\frac{dt\wedge ds}{(t-z)^3}] - g[\frac{dt\wedge ds}{(t-z)^2(s-z)}] + (\frac{a}{z}+\frac{b}{z-1})[\frac{dt\wedge ds}{(t-z)^2}] \\
&- (\frac{a}{z^2}+\frac{b}{(z-1)^2})[\frac{dt\wedge ds}{t-z}] + \frac{a}{z^2}[\frac{dt\wedge ds}{t}] + \frac{b}{(z-1)^2}[\frac{dt\wedge ds}{t-1}] \\
&= -c^{-1}(c-1)^{-1}w''' + c^{-1}(c-1)^{-1}(\frac{a}{z}+\frac{b}{z-1})\omega'' + c^{-1}(\frac{a}{z^2}+\frac{b}{(z-1)^2})\omega' \\
&- (3c+g)[\frac{dt\wedge ds}{(t-z)^2(s-z)}] - c(c-1)^{-1}(\frac{a}{z}+\frac{b}{z-1})[\frac{dt\wedge ds}{(t-z)(s-z)}] \\
&+ \frac{a}{z^2}[\frac{dt\wedge ds}{t}] + \frac{b}{(z-1)^2}[\frac{dt\wedge ds}{t-1}]
\end{aligned} \tag{1.2.14}$$

The last two terms may be expressed in terms of ω and its derivatives by writing

$$\begin{aligned}
0=\widetilde{d}[(t-1)ds] &= (1+a+b+c+\frac{g}{2})[dt\wedge ds] - a[\frac{dt\wedge ds}{t}] + c(z-1)[\frac{dt\wedge ds}{t-z}] \\
&= 2(1+a+b+c+\frac{g}{2})\omega - (z-1)\omega' - a[\frac{dt\wedge ds}{t}]
\end{aligned} \tag{1.2.15}$$

and

$$\begin{aligned}
0=\widetilde{d}[tds] &= (1+a+b+c+\frac{g}{2})[dt\wedge ds] + b[\frac{dt\wedge ds}{t-1}] + cz[\frac{dt\wedge ds}{t-z}] \\
&= 2(1+a+b+c+\frac{g}{2})\omega - z\omega' + b[\frac{dt\wedge ds}{t-1}]
\end{aligned} \tag{1.2.16}$$

To eliminate $[(t-z)^{-1}(s-z)^{-1}dt\wedge ds]$ from (1.2.14), we use

$$\begin{aligned}
0=\widetilde{d}[\frac{ds}{t-z}] &= (c-1)[\frac{dt\wedge ds}{(t-z)^2}] - \frac{g}{2}[\frac{dt\wedge ds}{(t-z)(s-z)}] + (\frac{a}{z}+\frac{b}{z-1})[\frac{dt\wedge ds}{t-z}] \\
&- \frac{a}{z}[\frac{dt\wedge ds}{t}] - \frac{b}{z-1}[\frac{dt\wedge ds}{t-1}] \\
&= c^{-1}\omega'' - c^{-1}(\frac{a}{z}+\frac{b}{z-1})\omega' \\
&- (c+\frac{g}{2})[\frac{dt\wedge ds}{(t-z)(s-z)}] - \frac{a}{z}[\frac{dt\wedge ds}{t}] - \frac{b}{z-1}[\frac{dt\wedge ds}{t-1}]
\end{aligned} \tag{1.2.17}$$

Finally, to eliminate $[(t-z)^{-2}(s-z)^{-1}dt\wedge ds]$ from (1.2.14), write

$$\begin{aligned}
0=\widetilde{d}[\frac{ds}{(t-z)(s-z)}] &= (c-1)[\frac{dt\wedge ds}{(t-z)^2(s-z)}] + \left(\frac{a}{z}+\frac{b}{z-1}\right)[\frac{dt\wedge ds}{(t-z)(s-z)}] \\
&- \frac{a}{z}[\frac{dt\wedge ds}{(t-z)s}] - \frac{b}{z-1}[\frac{dt\wedge ds}{(t-z)(s-1)}]
\end{aligned} \tag{1.2.18}$$

This introduces the additional terms $[(t-z)^{-1}s^{-1}dt\wedge ds]$ and $[(t-z)^{-1}(s-1)^{-1}dt\wedge ds]$ which may in turn be eliminated by using

$$0=\widetilde{d}[\frac{ds}{s-z}] = (c+\frac{g}{2})[\frac{dt\wedge ds}{(t-z)(s-z)}] + a[\frac{dt\wedge ds}{(t-z)s}] + b[\frac{dt\wedge ds}{(t-z)(s-1)}] \tag{1.2.19}$$

$$0=\widetilde{d}[(t-z)\frac{ds}{s-z}] = (1+a+b+c+g)[\frac{dt\wedge ds}{t-z}] - az[\frac{dt\wedge ds}{(t-z)s}] - b(z-1)[\frac{dt\wedge ds}{(t-z)(s-1)}] \tag{1.2.20}$$

A straightforward check now shows that, up to coboundaries, ω satisfies (1.1). To conclude, notice that if f is one of the differential forms of which we have taken the twisted differential, then

$$\int_{C_1\times C_2}\Phi\widetilde{d}f = \int_{C_1\times C_2} d(\Phi f) = \int_{\partial(C_1\times C_2)}\Phi f = 0 \tag{1.2.21}$$

since, by (1.1.4)–(1.1.5), Φf vanishes on $\partial(C_1 \times C_2) = C_1(\partial I) \times C_2 \cup C_2 \times C_2(\partial I)$. Thus if $P(\frac{d}{dz})$ is the Dotsenko–Fateev differential operator, $\int_{C_1\times C_2} P(\frac{d}{dz})\omega = 0$ and it follows that $\int_{C_1\times C_2}\omega$ satisfies the Dotsenko–Fateev equation since $\int \frac{d}{dz}\omega = \frac{d}{dz}\int\omega$. This is clear if the end–points of the C_i lie in $\{0,1,\infty\}$ and follow in general because Φ vanishes on $\partial(C_1\times C_2)$ by (1.1.4)–(1.1.5) $\diamond$

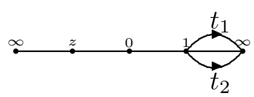
1.3. Monodromy properties of contour integrals.

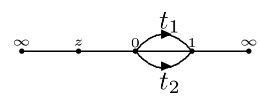
Following [**DF**], we show below how suitable choices of contours in the integral representation (1.6) lead to basis of solutions of the Dotsenko–Fateev equation diagonalising the monodromy at a given singular point $z_0 \in \{0,1,\infty\}$. We restrict our attention to $z_0 = 0$ and ∞ since these are the only cases we shall need. For graphical simplicity, the contours are represented with z lying on the negative real axis $(-\infty, 0)$. To fix conventions, all functions of the form w^λ where w is a function of t_1, t_2 will be defined on the w plane with a cut along $[-\infty, 0]$ so that $\arg w \in (-\pi, \pi)$ and

$$(-w)^\lambda = \begin{cases} w^\lambda e^{-i\pi\lambda} & \text{if } \operatorname{Im} w > 0 \\ w^\lambda e^{i\pi\lambda} & \text{if } \operatorname{Im} w < 0 \end{cases} \tag{1.3.1}$$

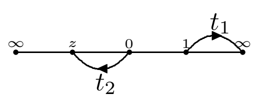
The complex powers z^λ however are defined with a cut along $z \in [0,\infty]$.

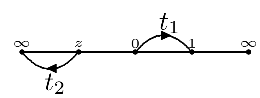
Proposition 1.3.1. *Let the parameters a, b, c, g of the Dotsenko–Fateev equation lie in the range* (1.1.4)–(1.1.5)*. Then the integrals* (1.6) *corresponding to the double contours*

$I_{0,1}$: 

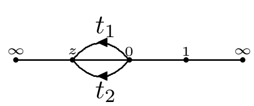
$I_{\infty,1}$: 

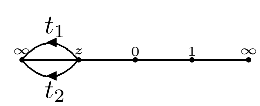
(1.3.2)

$I_{0,2}$: 

$I_{\infty,2}$: 

(1.3.3)

$I_{0,3}$: 

$I_{\infty,3}$: 

(1.3.4)

yield solutions having the following monodromy at $z = 0$ and $z = \infty$ respectively.

$$I_{0,1} = z^0(\rho_{0,1} + zO(z)) \qquad I_{\infty,1} = \left(\frac{1}{z}\right)^{-2c}(\rho_{\infty,1} + \frac{1}{z}O(\frac{1}{z})) \tag{1.3.5}$$

$$I_{0,2} = z^{1+a+c}(\rho_{0,2} + zO(z)) \qquad I_{\infty,2} = \left(\frac{1}{z}\right)^{-(1+a+b+2c+g)}(\rho_{\infty,2} + \frac{1}{z}O(\frac{1}{z})) \tag{1.3.6}$$

$$I_{0,3} = z^{2(1+a+c+\frac{g}{2})}(\rho_{0,3} + zO(z)) \qquad I_{\infty,3} = \left(\frac{1}{z}\right)^{-2(1+a+b+c+\frac{g}{2})}(\rho_{\infty,3} + \frac{1}{z}O(\frac{1}{z})) \tag{1.3.7}$$

The leading coefficients are given by

$$\rho_{0,1} = \frac{1}{2}(1+e^{i\pi g})J_2(-(2+a+b+c+g), b; g) \qquad \rho_{\infty,1} = \frac{1}{2}(1+e^{i\pi g})J_2(a,b;g) \tag{1.3.8}$$

$$\rho_{0,2} = B(a+1,c+1)B(-(1+a+b+c), b+1) \qquad \rho_{\infty,2} = B(a+1,b+1)B(-(1+a+b+c), c+1) \tag{1.3.9}$$

$$\rho_{0,3} = \frac{1}{2}(1+e^{i\pi g})J_2(a,c;g) \qquad \rho_{\infty,3} = \frac{1}{2}(1+e^{i\pi g})J_2(-(2+a+b+c+g), c; g) \tag{1.3.10}$$

where B is the Euler β function, and do not vanish if $g \neq -1$.

Proof. Consider $I_{\infty,1}$ first. The integrand in $I_{\infty,1}$ may be written, up to a complex phase factor as $z^{2c}\psi(z)$ where $\psi(z) = t_1^a(t_1-1)^b(1-t_1z^{-1})^c t_2^a(t_2-1)^b(1-t_2z^{-1})^c(t_1-t_2)^g$ is single–valued on $|z| > 1$ when $|t_1|, |t_2| < 1$. Thus, $I_{\infty,1} = \left(\frac{1}{z}\right)^{-2c}\Psi(\frac{1}{z})$ where Ψ is holomorphic and single–valued in a

neighborhood of 0. The leading term of Ψ may be computed by deforming successively the contours C_1, C_2 and using Selberg's formula

$$\begin{aligned}\Psi(0) &= \int_{C_1}\int_{C_2} t_1^a(t_1-1)^b t_2^a(t_2-1)^b(t_1-t_2)^g dt_1 dt_2\\ &= \int_0^1\int_{C_2} t_1^a(1-t_1)^b t_2^a(1-t_2)^b(t_1-t_2)^g dt_1 dt_2\\ &= \int_0^1\int_{C_2^1} t_1^a(1-t_1)^b t_2^a(1-t_2)^b(t_1-t_2)^g dt_1 dt_2 + \int_0^1\int_{C_2^2} t_1^a(1-t_1)^b t_2^a(1-t_2)^b(t_1-t_2)^g dt_1 dt_2\end{aligned} \tag{1.3.11}$$

where the C_2^1 and C_2^2 lie below the real axis and join 0 to t_1 and t_1 to 1 respectively. Since $\Re(t_1-t_2)>0$ along C_2^1 we may deform it onto the segment $(0,t_1)$ to get a contribution

$$\int_0^1\int_0^{t_1} t_1^a(1-t_1)^b t_2^a(1-t_2)^b|t_1-t_2|^g dt_1 dt_2 = \frac{1}{2}\int_0^1\int_0^1 t_1^a(1-t_1)^b t_2^a(1-t_2)^b|t_1-t_2|^g dt_1 dt_2 \tag{1.3.12}$$

On C_2^2, (1.3.1) gives $(t_1-t_2)^g = e^{i\pi g}(t_2-t_1)^g$. Deforming onto the real axis we get the contribution

$$e^{i\pi g}\int_0^1\int_{t_1}^1 t_1^a(1-t_1)^b t_2^a(1-t_2)^b|t_1-t_2|^g dt_1 dt_2 = e^{i\pi g}\frac{1}{2}\int_0^1\int_0^1 t_1^a(1-t_1)^b t_2^a(1-t_2)^b|t_1-t_2|^g dt_1 dt_2 \tag{1.3.13}$$

And therefore, $\Psi(0)=\frac{1}{2}(1+e^{i\pi g})J_2(a,b;g)$ which, in view of the remark following proposition 1.1.1 vanishes iff $g\in 2\mathbb{Z}+1$. However, by (1.1.4)–(1.1.5), $\Re g>-1$ and $\Re g<-1-\Re(a+b+c)<2$ and it follows that $\Psi(0)$ is non–zero if $g\neq 1$. $I_{\infty,3}$ is treated similarly by setting $t_i = zu_i^{-1}$, where the u_i lie on contours C_i' joining 0 to 1. Performing the change of variables, one finds

$$\begin{aligned}I_{\infty,3} =&\Big(\frac{1}{z}\Big)^{-2(1+a+b+c+\frac{g}{2})}\int_{C_1'}\int_{C_2'} du_1 du_2\\ &u_1^{-(2+a+b+c+g)}(1-u_1z^{-1})^b(1-u_1)^c u_2^{-(2+a+b+c+g)}(1-u_2z^{-1})^b(1-u_2)^c(u_1-u_2)^g\end{aligned} \tag{1.3.14}$$

The coefficient of the leading term at $z=\infty$ is obtained as for $I_{\infty,1}$ and is equal to $\frac{1}{2}(1+e^{i\pi g})J_2(-(2+a+b+c+g),c;g)$ so that it does not vanish if $g\neq 1$. Finally, $I_{\infty,2}$ is obtained by setting $t_2 = zu_2^{-1}$ where $u_2\in C_2'$ runs from 0 to 1. This yields

$$\begin{aligned}I_{2,\infty} =& -\Big(\frac{1}{z}\Big)^{-(1+a+b+2c+g)}\\ &\int_{C_1}\int_{C_2'} t_1^a(t_1-1)^b(1-t_1z^{-1})^c u_2^{-(2+a+b+c+g)}(1-u_2z^{-1})^b(1-u_2)^c(1-t_1u_2z^{-1})^g dt_1 du_2\end{aligned} \tag{1.3.15}$$

with leading coefficient

$$\int_{C_1} t_1^a(t_1-1)^b dt_1\int_{C_2'} u_2^{-(2+a+b+c+g)}(1-u_2)^c du_2 = B(a+1,b+1)B(-(1+a+b+c+g),c+1)\neq 0 \tag{1.3.16}$$

The cases $I_{0,i}$, $i=1\dots 3$ follow similarly $\diamond$

1.4. Connection matrices for the Dotsenko–Fateev equation.

We compute below the analytic continuation of the solution $I_{0,1}$ of the Dotsenko–Fateev equation equation from 0 to ∞ and re–express it in terms of the solutions $I_{\infty,i}$ diagonalising the monodromy at ∞. Following [**DF**], we use the technique of contour deformation. This consists in deforming the double contour corresponding to $I_{0,1}$ in various ways. Combinining these with appropriate phase corrections, one obtains a linear combination of contours corresponding to the solutions $I_{\infty,i}$. The same method applies to all solutions $I_{0,i}$ and works equally well to compute the analytic continuation of solutions from 0 to 1, see [**DF**]. For later use however, we only need the results pertaining to $I_{0,1}$.

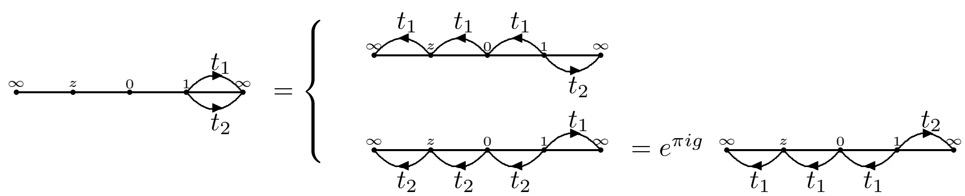
We shall compute the analytic continuation along the negative real axis and therefore assume that $z \in (-\infty, 0)$. We adhere to the notations and conventions of §1.3.

PROPOSITION 1.4.1. *Let $I_{z_i,j}$ be the solutions of the Dotsenko–Fateev equation given by proposition 1.3.1. Then*

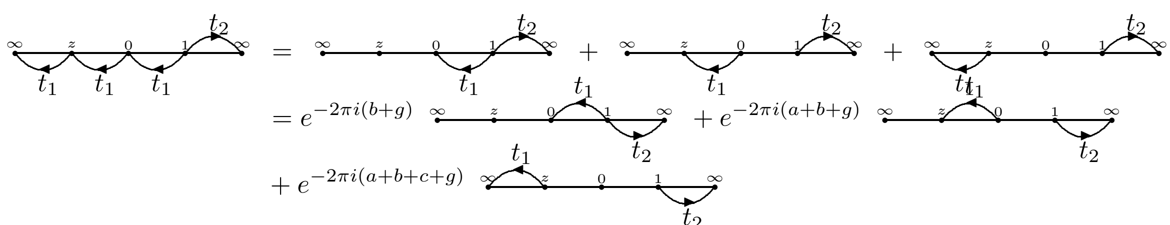
$$\begin{aligned} I_{0,1} &= \frac{\mathrm{s}(a)\,\mathrm{s}(a+\frac{g}{2})}{\mathrm{s}(a+b+\frac{g}{2})\,\mathrm{s}(a+b+g)} I_{\infty,1} \\ &\quad + 2e^{-\pi i(a+c+\frac{g}{2})}\,\mathrm{c}(\frac{g}{2}) \frac{\mathrm{s}(a)\,\mathrm{s}(c)}{\mathrm{s}(a+b)\,\mathrm{s}(a+b+g)} I_{\infty,2} \\ &\quad + \frac{\mathrm{s}(c)\,\mathrm{s}(c+\frac{g}{2})}{\mathrm{s}(a+b+\frac{g}{2})\,\mathrm{s}(a+b+\frac{g}{2})} I_{\infty,3} \end{aligned} \tag{1.4.1}$$

where $\mathrm{s}(\alpha) = \sin(\pi\alpha)$ *and* $\mathrm{c}(\alpha) = \cos(\pi\alpha)$.

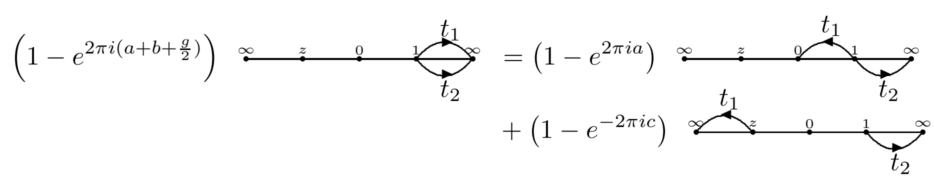
PROOF. If we take the equator of $\mathbb{P}^1$ as the line through $0, 1, \infty$ and z, then deforming the t_1 contour in the northern hemisphere or the t_2 contour in the southern one gives



where the last equality follows from the change of variables $t_1 \to t_2$, $t_2 \to t_1$ (so that $(t_1 - t_2)^g \to (t_2 - t_1)^g = e^{\pi i g}(t_1 - t_2)^g$ by (1.3.1) since since $\mathrm{Im}(t_1 - t_2) < 0$). Now

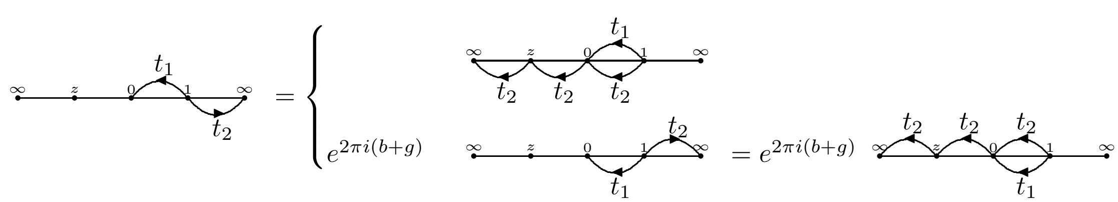


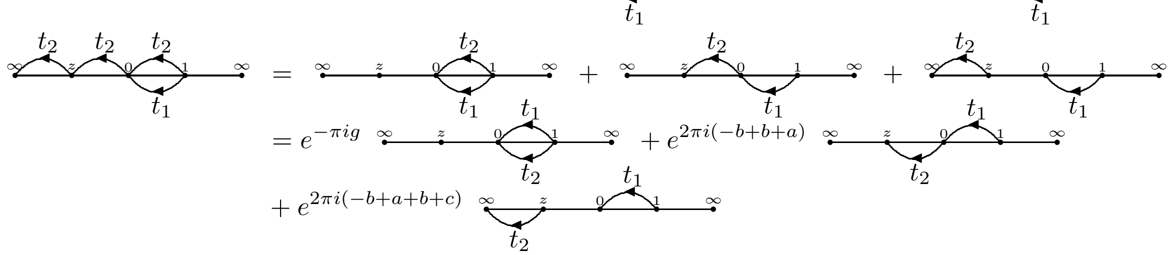
and therefore



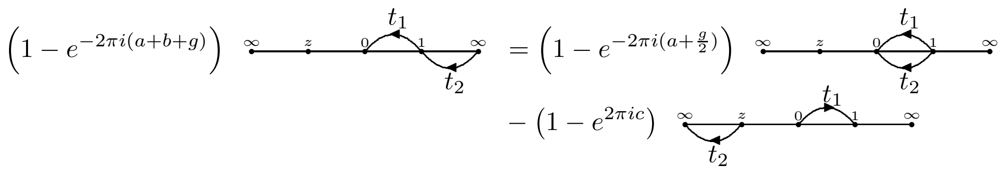
(1.4.2)

Proceeding in a similar fashion, we may deform the first summand on the right hand–side of (1.4.2)



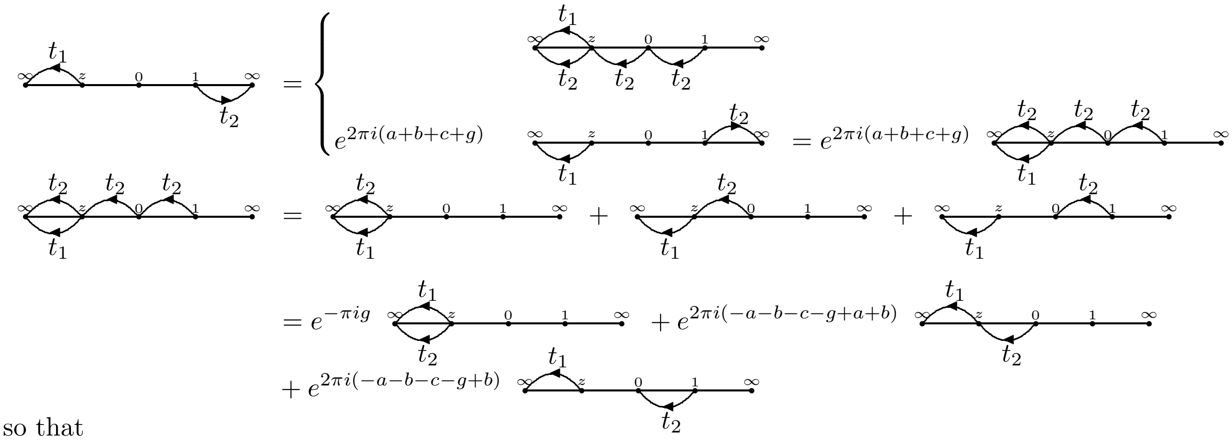


and get

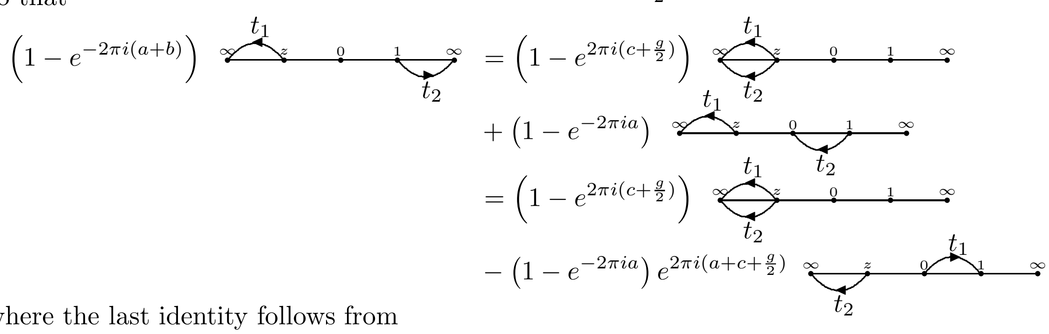


(1.4.3)

The second summand on the other hand yields

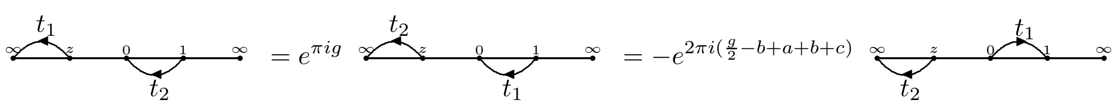


so that



(1.4.4)

where the last identity follows from



Substituting (1.4.3) and (1.4.4) in (1.4.2) gives finally

$$\begin{aligned}
&\left(1-e^{2\pi i(a+b+\frac{g}{2})}\right)\left(1-e^{-2\pi i(a+b+g)}\right)\left(1-e^{-2\pi i(a+b)}\right)I_1^0 = \\
&\left(1-e^{2\pi ia}\right)\left(1-e^{-2\pi i(a+b)}\right)\left(1-e^{-2\pi i(a+\frac{g}{2})}\right)I_1^\infty - \\
&\Big(\left(1-e^{2\pi ia}\right)\left(1-e^{-2\pi i(a+b)}\right)\left(1-e^{2\pi ic}\right)e^{-2\pi i(a+c+\frac{g}{2})} + \\
&\left(1-e^{-2\pi ic}\right)\left(1-e^{-2\pi i(a+b+g)}\right)\left(1-e^{-2\pi ia}\right)\Big)I_2^\infty + \\
&\left(1-e^{-2\pi ic}\right)\left(1-e^{-2\pi i(a+b+g)}\right)\left(1-e^{2\pi i(c+\frac{g}{2})}\right)I_3^\infty
\end{aligned}$$

which, after simplification, yields (1.4.1) $\diamond$

2. The Knizhnik–Zamolodchikov equations with values in $(V_{k\theta_1}\otimes V_{k\theta_1}^*\otimes V_{\theta_1}\otimes V_{\theta_1}^*)^{\mathrm{SO}_{2n}}$

In the rest of this chapter, we label representations of Spin_{2n} by their highest weight. Thus, V_{θ_1} and $V_{k\theta_1}$ denote the vector representation of SO_{2n} and its k–fold symmetric, traceless power respectively.

In §2.2, we compute explicitly the residue matrices Ω_{ij} of the Knizhnik–Zamolodchikov equations corresponding to the tensor product $(V_{k\theta_1} \otimes V^*_{k\theta_1} \otimes V_{\theta_1} \otimes V^*_{\theta_1})^{\mathrm{SO}_{2n}}$. The calculation depends on some formulae of Christe and Flume [**CF**] which are obtained in §2.1.

2.1. The Ω_{ij} matrices for $(V_4 \otimes V_3^* \otimes V_2^* \otimes V_1^*)^G$.

Let G be a compact, connected simple Lie group. Recall from chapter VII that to any tensor product $(V_n \otimes V^*_{n-1} \otimes \cdots \otimes V_1^*)^G$ of irreducible G–modules one associates the endomorphisms $\Omega_{ij} = \pi_i(X_k)\pi_j(X^k)$, $1 \le i, j \le n$ where X_k, X^k are basis of $\mathfrak{g}_{\mathbb{C}}$ dual with respect to the basic inner product. We compute below some matrix entries of the Ω_{ij} when $n = 4$, in any tensor product basis corresponding to the the isomorphism

$$(V_4 \otimes V_3^* \otimes V_2^* \otimes V_1^*)^G \cong \bigoplus_U \mathrm{Hom}_G(U, V_3^* \otimes V_4) \otimes \mathrm{Hom}_G(V_1 \otimes V_2, U) \tag{2.1.1}$$

where U ranges over the irreducible summands of $V_1 \otimes V_2$. The inner product is then given by $(\psi_U \otimes \phi_U, \psi_W \otimes \phi_W) = \mathrm{tr}_{V_1 \otimes V_2}(\phi_W^* \psi_W^* \psi_U \phi_U) = \mathrm{tr}_W(\psi_W^* \psi_U \phi_U \phi_W^*)$. The following formulae are somewhat imprecisely stated in [**CF**]

PROPOSITION 2.1.1. *Let Ω_{12}, Ω_{23} be the operators corresponding to the tensor product $(V_4 \otimes V_3^* \otimes V_2^* \otimes V_1^*)^G$. Then*

(i) *Ω_{12} is diagonal in any basis corresponding to the decomposition* (2.1.1). *Its diagonal entries are*
$$(\Omega_{12})_{\psi_U \otimes \phi_U, \psi_U \otimes \phi_U} = \frac{1}{2}(C_U - C_1 - C_2) =: \delta_U \tag{2.1.2}$$
(ii) *If $U = \mathbb{C}$ is the trivial representation, then*
$$(\Omega_{23})_{\psi_U \otimes \phi_U, \psi_U \otimes \phi_U} = 0 \tag{2.1.3}$$
(iii) *If $\mathrm{Hom}_G(U \otimes \mathfrak{g}_{\mathbb{C}}, U) = \mathbb{C}$, so that $U \ne 0$ and $C_U > 0$ then*
$$(\Omega_{23})_{\psi_U \otimes \phi_U, \psi_U \otimes \phi_U} = -\frac{1}{4C_U}(C_U + C_3 - C_4)(C_U + C_2 - C_1) \tag{2.1.4}$$
(iv) *If $\mathrm{Hom}_G(U \otimes \mathfrak{g}_{\mathbb{C}}, U') = 0$, then*
$$(\Omega_{23})_{\psi_U \otimes \phi_U, \psi_{U'} \otimes \phi_{U'}} = (\Omega_{23})_{\psi_{U'} \otimes \phi_{U'}, \psi_U \otimes \phi_U} = 0 \tag{2.1.5}$$

PROOF. (i) From the G-equivariance of ϕ_U, we get

$$\begin{aligned}\Omega_{12}\psi_U \otimes \phi_U &= \psi_U \phi_U \pi_2(X_k)\pi_1(X^k) \\ &= \frac{1}{2}\psi_U\phi_U(\pi_2(X_k) + \pi_1(X_k))(\pi_2(X^k) + \pi_1(X^k)) \\ &\quad - \frac{1}{2}\psi_U\phi_U(\pi_2(X_k)\pi_2(X^k) + \pi_1(X_k)\pi_1(X^k)) \\ &= \frac{1}{2}\psi_U \pi_U(X_k)\pi_U(X^k)\phi_U - \frac{1}{2}(C_1 + C_2)\psi_U\phi_U \\ &= \frac{1}{2}(C_U - C_1 - C_2)\psi_U \otimes \pi_U\end{aligned} \tag{2.1.6}$$

(ii) The map $\mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$, $X \to \phi_U \pi_2(X)\phi_U^*$ is G-invariant and therefore zero. Thus

$$(\Omega_{23}\psi_U \otimes \phi_U, \psi_U \otimes \phi_U) = -\,\mathrm{tr}_U(\psi_U^* \pi_{\overline{3}}(X_k)\psi_U\phi_U\pi_2(X^k)\phi_U^*) = 0 \tag{2.1.7}$$

(iii) The map $\mathfrak{g}_{\mathbb{C}} \otimes U \to U$, $X \otimes u \to \phi_U \pi_2(X)\phi_U^* u$ commutes with G and is therefore proportional to $u \otimes X \to \pi_U(X)u$. Similarly, $\psi_U^* \pi_{\overline{3}}(X)\psi_U = c_\psi \pi_U(X)$ for any $X \in \mathfrak{g}$. It follows that

$$(\Omega_{23}\psi_U \otimes \phi_U, \psi_U \otimes \phi_U) = -\,\mathrm{tr}_U(\psi_U^* \pi_{\overline{3}}(X_k)\psi_U\phi_U\pi_2(X^k)\phi_U^*) = -c_\psi c_\phi \dim(U) C_U \tag{2.1.8}$$

To evaluate the constants c_ψ, c_ϕ, write

$$\pi_U(X^k)\psi_U^*\pi_{\overline{3}}(X_k)\psi_U = c_\psi \pi_U(X^k)\pi_U(X_k) \tag{2.1.9}$$

The right hand–side yields $c_\psi C_U$ while the left hand–side gives, as in (i) $\frac{1}{2}(C_4 - C_U - C_3)\psi_U^*\psi_U$. The computation of c_ϕ is similar. Summarising,

$$\begin{aligned}(\Omega_{23}\psi_U \otimes \phi_U, \psi_U \otimes \phi_U) &= -\frac{1}{4C_U}(C_U + C_3 - C_4)(C_U + C_2 - C_1)\dim(U)\psi_U^*\psi_U\phi_U\phi_U^* \\ &= -\frac{1}{4C_U}(C_U + C_3 - C_4)(C_U + C_2 - C_1)\|\psi_U \otimes \phi_U\|^2\end{aligned} \tag{2.1.10}$$

(iv) The map $\mathfrak{g}_{\mathbb{C}} \otimes U \to U'$, $X \otimes u \to \psi_{U'}^*\pi_{\overline{3}}(X)\psi_U u$ commutes with G and therefore vanishes. It follows that

$$(\Omega_{23}\psi_U \otimes \phi_U, \psi_{U'} \otimes \phi_{U'}) = -\operatorname{tr}_{U'}(\psi_{U'}^*\pi_{\overline{3}}(X_k)\psi_U\phi_U\pi_2(X^k)\phi_{U'}^*) = 0 \tag{2.1.11}$$

The second identity follows by permuting U and U' and noticing that the adjoint representation is real and therefore self-dual so that

$$\operatorname{Hom}_G(U' \otimes \mathfrak{g}_{\mathbb{C}}, U) = \operatorname{Hom}_G(U', U \otimes \mathfrak{g}_{\mathbb{C}}) = \operatorname{Hom}_G(U \otimes \mathfrak{g}_{\mathbb{C}}, U')^* \tag{2.1.12}$$

$\diamond$

Remark. Clearly Ω_{13} and Ω_{23} are diagonal in any tensor product basis of $W = (V_4 \otimes V_3^* \otimes V_2^* \otimes V_1^*)^G$ corresponding respectively to the isomorphisms

$$W \cong \bigoplus_U \operatorname{Hom}_G(U, V_2^* \otimes V_4) \otimes \operatorname{Hom}_G(V_1 \otimes V_3, U) \tag{2.1.13}$$

and

$$W \cong \bigoplus_U \operatorname{Hom}_G(U, V_1^* \otimes V_4) \otimes \operatorname{Hom}_G(V_2 \otimes V_3, U) \tag{2.1.14}$$

In particular, their eigenvalues may be labelled by the irreducible summands of $V_1 \otimes V_3$ and $V_2 \otimes V_3$ respectively.

2.2. The Ω_{ij} matrices for $(V_{k\theta_1} \otimes V_{k\theta_1}^* \otimes V_{\theta_1} \otimes V_{\theta_1}^*)^{\mathrm{SO}_{2n}}$.

Specialising the results of §2.1 to $G = \mathrm{Spin}_{2n}$, we find

Proposition 2.2.1. *Let Ω_{ij} be the operators corresponding to $W = (V_{k\theta_1} \otimes V_{k\theta_1}^* \otimes V_{\theta_1} \otimes V_{\theta_1}^*)^{\mathrm{SO}_{2n}}$. Then*

(i) *The eigenvalues of Ω_{12} are labelled by the summands of*

$$V_{\theta_1} \otimes V_{\theta_1}^* = V_{2\theta_1} \oplus V_{\theta_1+\theta_2} \oplus V_0 \tag{2.2.1}$$

and are given by

$$\delta_{2\theta_1} = 1 \qquad \delta_{\theta_1+\theta_2} = -1 \qquad \delta_0 = -2n+1 \tag{2.2.2}$$

(ii) *Ω_{13} and Ω_{23} are conjugate. Their eigenvalues are labelled by the summands of*

$$V_{\theta_1} \otimes V_{k\theta_1} \cong V_{\theta_1}^* \otimes V_{k\theta_1} = V_{(k+1)\theta_1} \oplus V_{k\theta_1+\theta_2} \oplus V_{(k-1)\theta_1} \tag{2.2.3}$$

and are given by

$$\widetilde{\delta}_{(k+1)\theta_1} = k \qquad \widetilde{\delta}_{k\theta_1+\theta_2} = -1 \qquad \widetilde{\delta}_{(k-1)\theta_1} = -2(n-1) - k \tag{2.2.4}$$

(iii) *In the basis corresponding to*

$$W \cong \bigoplus_U \operatorname{Hom}_{\mathrm{SO}_{2n}}(U, V_{k\theta_1} \otimes V_{k\theta_1}^*) \otimes \operatorname{Hom}_{\mathrm{SO}_{2n}}(V_{\theta_1} \otimes V_{\theta_1}^*, U) \tag{2.2.5}$$

where U ranges over the summands of (2.2.1), *Ω_{12}, Ω_{23} and Ω_{13} are given by*

$$\Omega_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2n+1 \end{pmatrix} \quad \Omega_{23} = \begin{pmatrix} -n & x & 0 \\ \widetilde{x} & -n+1 & y \\ 0 & \widetilde{y} & 0 \end{pmatrix} \quad \Omega_{13} = \begin{pmatrix} -n & -x & 0 \\ -\widetilde{x} & -n+1 & -y \\ 0 & -\widetilde{y} & 0 \end{pmatrix} \tag{2.2.6}$$

where

$$x\widetilde{x}=\frac{(n-1)}{n}(k^2+2k(n-1)+n(n-2)) \qquad \text{and} \qquad y\widetilde{y}=\frac{k}{n}(k+2(n-1)) \tag{2.2.7}$$

PROOF. Let $V=V_{\theta_1}\cong V_{\theta_1}^*$ be the defining representation of SO_{2n}. V is minimal with weights $\pm\theta_j$, $1\leq j\leq n$ and therefore (2.2.1) and (2.2.3) follow from proposition I.2.2.2. In particular,

$$W\cong \mathrm{End}_{\mathrm{SO}_{2n}}(V\otimes V_{k\theta_1})\cong\mathbb{C}^3 \tag{2.2.8}$$

Since the Casimir C_λ of a representation of highest weight λ is $\langle\lambda,\lambda+2\rho\rangle$ where $2\rho=2\sum_j(n-j)\theta_j$ is the sum of the positive roots of SO_{2n}, the Casimirs of the summands of (2.2.1) are $C_{2\theta_1}=4n$, $C_{\theta_1+\theta_2}=4(n-1)$ and $C_0=0$. It follows from proposition 2.1.1 and $C_{\theta_1}=2n-1$ that the eigenvalues of Ω_{12} are $1,-1,-2n+1$. We claim that each has multiplicity one, *i.e.* that the spaces $\mathrm{Hom}_{\mathrm{SO}_{2n}}(U,V_{k\theta_1}^*\otimes V_{k\theta_1})$ are one-dimensional for $U=V_{2\theta_1}$, $U=V_0\cong\mathbb{C}$ and $U=V_{\theta_1+\theta_2}\cong\mathfrak{so}_{2n}$ so that Ω_{12} is given by 2.2.1. Since the Hom spaces are non–zero in the last two cases and W is three–dimensional, we will prove our claim by exhibiting a non–zero intertwiner $V_{k\theta_1}\otimes V_{2\theta_1}\to V_{k\theta_1}$. To this end, realise $V_{k\theta_1}$ as the space of traceless, symmetric k-tensors as follows. Let $S^kV\subset V^{\otimes k}$ be the k-fold symmetric tensor power of V spanned by the tensors

$$[v_1\otimes\cdots\otimes v_k]=\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)} \tag{2.2.9}$$

Let moreover B be the SO_{2n}–invariant symmetric, bilinear form on V. The natural trace map

$$\tau_k:V^{\otimes k}\to V^{\otimes(k-2)},\ v_1\otimes\cdots\otimes v_k\to\sum_{1\leq i<j\leq k}B(v_i,v_j)v_1\otimes\cdots\otimes\widehat{v_i}\otimes\cdots\otimes\widehat{v_j}\otimes\cdots\otimes v_k \tag{2.2.10}$$

satisfies $\tau_k[v_1\otimes\cdots\otimes v_k]=[\tau_k(v_1\otimes\cdots\otimes v_k)]$ and therefore restricts to a map $\tau_k:S^kV\to S^{k-2}V$. The kernel of $\tau_k|_{S^kV}$ is $V_{k\theta_1}$ and the intertwiner $V_{k\theta_1}\otimes V_{2\theta_1}\to V_{k\theta_1}$ is given by $[v_1\otimes\cdots\otimes v_k]\otimes[w_1\otimes w_2]\to P\tau_{k+2}[v_1\otimes\cdots\otimes v_k\otimes w_1\otimes w_2]$ where $P:S^kV\to V_{k\theta_1}$ is the orthogonal projection.

Let now $i:V_{\theta_1}\to V_{\theta_1}^*$ be the SO_{2n} identification and σ the permutation on $V_{\theta_1}\otimes V_{\theta_1}$. Then, $\Omega_{13}=F\Omega_{23}F^{-1}$ where $F=(1\otimes i)\sigma(1\otimes i^{-1})$. Notice that σ acts as multiplication by $1,-1,1$ respectively on the summands of (2.2.1) and therefore

$$F=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix} \tag{2.2.11}$$

To compute Ω_{23}, notice that the Casimirs of the summands of (2.2.3) are $C_{(k+1)\theta_1}=k(k+2(n-1))$, $C_{k\theta_1+\theta_2}=k^2+2(k+1)(n-1)-1$ and $C_{(k-1)\theta_1}=(k-1)(k-1+2(n-1))$. Since Ω_{23} acts by pre-multiplication on $V_{k\theta_1}\otimes V^*$ when W is described as

$$\mathrm{End}_{\mathrm{SO}_{2n}}(V_{k\theta_1}\otimes V^*)\cong\bigoplus_U\mathrm{Hom}_{\mathrm{SO}_{2n}}(U,V_{k\theta_1}\otimes V^*)\otimes\mathrm{Hom}_{\mathrm{SO}_{2n}}(V_{k\theta_1}\otimes V^*,U) \tag{2.2.12}$$

it is diagonal in the basis given by the orthogonal projections onto the irreducible summands of $V_{k\theta_1}\otimes V^*$. The corresponding eigenvalues are, by (i) of proposition 2.1.1, $k,-1,-2(n-1)-k$. Finally, by the Brauer rules for computing tensor products [**Hu**, §24.4, Ex. 9] and the fact that the roots of $\mathfrak{so}_{2n}$ are $\pm(\theta_i\pm\theta_j)$, $1\leq i<j\leq n$, we find $\mathrm{Hom}_{\mathrm{SO}_{2n}}(V_{2\theta_1}\otimes\mathfrak{so}_{2n},V_{2\theta_1})\cong\mathbb{C}$ and therefore, by (ii) and (iii) of proposition 2.1.1, we find $(\Omega_{23})_{2\theta_1,2\theta_1}=-n$, $(\Omega_{23})_{0,0}=0$ and, by (iv), $(\Omega_{23})_{2\theta_1,0}=(\Omega_{23})_{0,2\theta_1}=0$. The remaining entries are simply found by requiring that

$$\det(\Omega_{23}-t)=(k-t)(-1-t)(-2(n-1)-k-t) \tag{2.2.13}$$

The matrix giving Ω_{13} now follows from $\Omega_{13}=F\Omega_{23}F^{-1}$ and (2.2.11) $\diamond$

For later reference, we shall need the following immediate

COROLLARY 2.2.2. *Let $\kappa=\ell+2(n-1)$ and Ω_{ij} be the matrices given by proposition 2.2.1 where $1\leq k\leq\ell$. Then*

(i) *The eigenvalues of* $\kappa^{-1}\Omega_{12}$ *do not differ by integers unless* $\ell = 2$. *For* $\ell = 2$, $\kappa^{-1}(\delta_{2\theta_1} - \delta_0) = 1$ *and all other eigenvalues do not differ by integers.*

(ii) *The eigenvalues of* $\kappa^{-1}\Omega_{13}$ *do not differ by integers unless* $\ell = 2k$. *For* $\ell = 2k$, $\kappa^{-1}(\widetilde{\delta}_{(k+1)\theta_1} - \widetilde{\delta}_{(k-1)\theta_1}) = 1$ *and all other eigenvalues do not differ by integers.*

3. The braiding relations

3.1. General strategy.

We consider in this section the level ℓ Knizhnik–Zamolodchikov equations

$$\frac{dF}{dz} = \frac{1}{\kappa}\Big(\frac{\Omega_{12} - \delta_0}{z} + \frac{\Omega_{23}}{z-1}\Big)F \tag{3.1.1}$$

with values in $(V_{k\theta_1} \otimes V^*_{k\theta_1} \otimes V_{\theta_1} \otimes V^*_{\theta_1})^{\mathrm{SO}_{2n}}$, where $\kappa = \ell + \frac{C_{\mathfrak{g}}}{2}$, with $\frac{C_{\mathfrak{g}}}{2} = 2(n-1)$ the dual Coxeter number of SO_{2n} and the matrices Ω_{ij} and δ_0 are given by proposition 2.2.1.

We prove in §3.3 that the reduction of (3.1.1) to a scalar, third order equation coincides with the Dotsenko–Fateev equation (1.1) for a suitable choice of the parameters a, b, c, g and use this to derive the analytic continuation of solutions of (3.1.1) from 0 to ∞ via proposition 1.4.1. These calculations are used in §3.4 to obtain the main result of this chapter, namely that the structure constants governing the braiding of the primary fields with charges V_{θ_1} and $V_{k\theta_1}$ do not vanish. A number of technical difficulties which we describe below arise in this approach. We circumvent them in §3.2.

Unfortunately, the values of the parameters of the Dotsenko–Fateev equation (1.1) for which it coincides with the scalar reduction of (3.1.1) lie outside the range (1.1.4)–(1.1.5) where the solutions of (1.1) may be expressed by means of generalised Euler integrals. A further difficulty arises in identifying specific solutions of (3.1.1) with those of (1.1). This needs to be done with great care for we are ultimately concerned with the analytic continuation of a given solution of (3.1.1), namely the reduced four–point function of the product of primary fields with charges V_{θ_1} and $V_{k\theta_1}$. For generic values of the level $\ell \in \mathbb{N}$, the identification is easily obtained by matching the monodromic behaviour of the solutions about a given singular point. However, when $\ell \in \{2, 2k\}$, corollary 2.2.2 implies that the monodromy operator corresponding to (3.1.1) has degenerate eigenvalues at 0 or ∞. For these values of ℓ, the solutions of (3.1.1) are not uniquely characterised by their monodromy about these singular points and the identification becomes ambiguous.

We shall circumvent both of the above difficulties in the following way. We prove in §3.2 that four–point functions at a given level $\ell = \ell_0$ may be analytically continued in $\ell \in (\mathbb{C}\setminus\mathbb{R}) \cup D$ trough solutions of the Knizhnik–Zamolodchikov equations where D is a disc centred at ℓ_0, so that the braiding coefficients depend holomorphically on ℓ [1]. When ℓ is not real, the eigenvalues of the residue matrices of the Knizhnik–Zamolodchikov equations do not differ by integers and the identification of the continued four–point functions with solutions of the Dotsenko–Fateev equation is unambiguous. Moreover, if ℓ has a large negative real part, the parameters of the Dotsenko–Fateev equation fall into the good range (1.1.4)–(1.1.5) and the analytic continuation from 0 to ∞ in the z variable may therefore be computed. This yields the braiding coefficients for complex ℓ. The required values are obtained by evaluation at $\ell = \ell_0$.

3.2. Holomorphic dependence of four–point functions on the level.

Let G be a compact, connected, simple Lie group with dual Coxeter number $\frac{C_{\mathfrak{g}}}{2}$.

LEMMA 3.2.1. *Let* ϕ_3, ϕ_2 *be primary fields at level* ℓ_0 *with vertices* $\begin{pmatrix} V_3 \\ V_4\ U \end{pmatrix}$, $\begin{pmatrix} V_2 \\ U\ V_1 \end{pmatrix}$ *and* $F = \sum_{n\geq 0} a_n z^n$ *the formal power series expansion of the corresponding reduced four–point function. If*

[1] As will become apparent in §3.2 this is not true of general solutions of the Knizhnik–Zamolodchikov equations.

$\Omega_{12}(a,b)$ is the subspace of $V = (V_4 \otimes V_3^* \otimes V_2^* \otimes V_1^*)^G$ corresponding to the Ω_{12}-eigenvalues $\delta \in (a,b)$, then for $n \geq 1$

$$a_n \in \Omega_{12}(-\infty, \delta_U + n(\ell_0 + \frac{C_{\mathfrak{g}}}{2})) \tag{3.2.1}$$

Proof. By proposition 2.1.1, Ω_{12} acts diagonally in the basis of $V \cong \mathrm{Hom}_G(V_1 \otimes V_2, V_3^* \otimes V_4)$ given by the elements $\psi_W \otimes \phi_W \in \mathrm{Hom}_G(W, V_3^* \otimes V_4) \otimes \mathrm{Hom}_G(V_1 \otimes V_2, W)$, with corresponding eigenvalue $\delta_W = \frac{1}{2}(C_W - C_1 - C_2)$. Since $a_n = \phi_3(\cdot, n)\phi_2(\cdot, -n)$, we find

$$(a_n, \psi_W \otimes \phi_W) = \mathrm{tr}_{V_1 \otimes V_2}(\phi_W^* \psi_W^* a_n) = \mathrm{tr}_W(\psi_W^* \phi_3(\cdot, n)\phi_2(\cdot, -n)\phi_W^*) \tag{3.2.2}$$

Now, $\phi_2(\cdot, -n)\phi_W^*$ is a G-intertwiner $W \rightarrow \mathcal{H}_U(n)$ and is therefore zero unless $W \subset \mathcal{H}_U(n)$. By proposition 11.4 b) of [**Ka1**], the latter condition implies that $C_W - C_U < 2n(\ell_0 + \frac{C_{\mathfrak{g}}}{2})$ and therefore $(a_n, \psi_W \otimes \phi_W) = 0$ if $\delta_W - \delta_U = \frac{1}{2}(C_W - C_U) \geq n(\ell_0 + \frac{C_{\mathfrak{g}}}{2})$ $\diamond$

Remark. When expressed in terms of the power series expansion $F = \sum_n a_n z^n$ of a reduced four point-function with intermediate representation U, the Knizhnik–Zamolodchikov equation satisfied by F becomes the recurrence relation

$$(\Omega_{12} - \delta_U - n(\ell_0 + \frac{C_{\mathfrak{g}}}{2}))a_n = \Omega_{23} \sum_{m=0}^{n-1} a_m \tag{3.2.3}$$

A priori, this does not determine F completely from a_0 since $(\Omega_{12} - \delta_U - n(\ell_0 + \frac{C_{\mathfrak{g}}}{2}))$ may fail to be invertible for some n. The above corollary however shows that this is not the case since $a_n \in \mathrm{Ker}(\Omega_{12} - (\delta_U + n\kappa))^{\perp}$.

Proposition 3.2.2. *Let $\phi_2 : \mathcal{H}_{V_1}^{\mathrm{fin}} \otimes V_2[z, z^{-1}] \rightarrow \mathcal{H}_U^{\mathrm{fin}}$, $\phi_3 : \mathcal{H}_U^{\mathrm{fin}} \otimes V_3[z, z^{-1}] \rightarrow \mathcal{H}_{V_4}^{\mathrm{fin}}$ be primary fields at level $\ell_0 \in \mathbb{N}$ with initial terms Φ_2, Φ_3 and denote by F_{ℓ_0} the corresponding reduced four–point function with values in $V = (V_4 \otimes V_3^* \otimes V_2^* \otimes V_1^*)^G$. If the eigenvalues of $(\ell_0 + \frac{C_{\mathfrak{g}}}{2})^{-1}\Omega_{12}$ on V do not differ by integers larger than 1, there exists an $\epsilon > 0$ such that*

(i) *For any $\ell \in \mathcal{D} = \{\lambda | \operatorname{Im}\lambda \neq 0\} \cup \{\lambda | \, |\lambda - \ell_0| < \epsilon\}$ with $\ell \neq \ell_0$, the Knizhnik–Zamolodchikov equation*

$$(\ell + \frac{C_{\mathfrak{g}}}{2})\frac{dF_\ell}{dz} = \Big(\frac{\Omega_{12} - \delta_U}{z} + \frac{\Omega_{23}}{z-1}\Big)F_\ell \tag{3.2.4}$$

possesses a unique formal power series solution $F_\ell = \sum_{n \geq 0} a_n(\ell) z^n$ with $a_0(\ell) = \Phi_3\Phi_2$.

(ii) *F_ℓ is holomorphic on $\mathbb{C} \setminus [1, \infty)$ and the assignement $(\ell, z) \rightarrow F_\ell(z)$ defines a holomorphic function on $\mathcal{D} \times (\mathbb{C} \setminus [1, \infty))$.*

Proof. For any $\ell \in \mathbb{C}$, a formal power series $F_\ell = \sum_{n \geq 0} a_n(\ell) z^n$ satisfies (3.2.4) if, and only if

$$\Omega_{12} a_0(\ell) = \delta_U a_0(\ell) \tag{3.2.5}$$

$$(\Omega_{12} - \delta_U - n(\ell + \frac{C_{\mathfrak{g}}}{2}))a_n(\ell) = \Omega_{23} \sum_{m=0}^{n-1} a_m(\ell) \tag{3.2.6}$$

When $\ell = \ell_0$ and F_{ℓ_0} is the reduced four–point function of the product $\phi_3\phi_2$ then $a_0(\ell_0) = \Phi_3\Phi_2$ and (3.2.1) yields $a_1(\ell_0) \in \Omega_{12}(-\infty, \delta_U + \ell_0 + \frac{C_{\mathfrak{g}}}{2})$. Thus, by (3.2.6) with $n = 1$

$$\Omega_{23}\Phi_3\Phi_2 \in \Omega_{12}(-\infty, \delta_U + \ell_0 + \frac{C_{\mathfrak{g}}}{2}) \tag{3.2.7}$$

Let now $\epsilon > 0$ be such that the eigenvalues of $(\ell + \frac{C_{\mathfrak{g}}}{2})^{-1}\Omega_{12}$ do not differ by positive integers whenever $0 < |\ell - \ell_0| < \epsilon$ and set $\mathcal{D} = (\mathbb{C} \setminus \mathbb{R}) \cup \{\ell | \, |\ell - \ell_0| < \epsilon\}$. Then, for any $\ell \in \mathcal{D} \setminus \{\ell_0\}$, the matrices $(\Omega_{12} - \delta_U - n(\ell + \frac{C_{\mathfrak{g}}}{2}))$ are invertible and therefore (3.2.4) possesses a unique formal power series solution $F_\ell = \sum_{n \geq 0} a_n(\ell) z^n$ with $a_0(\ell) = \Phi_3\Phi_2$. By (3.2.6), the $a_n(\ell)$ are rational functions of ℓ and are holomorphic on $\mathcal{D} \setminus \{\ell_0\}$. We claim that $a_n(\ell) \rightarrow a_n(\ell_0)$ as $\ell \rightarrow \ell_0$ so that ℓ_0 is a removable singularity of the $a_n(\ell)$. By our assumption on the eigenvalues of Ω_{12} we need only check this for $n = 1$. In this case, the result follows from (3.2.7) since $a_1(\ell) = (\Omega_{12} - \delta_U - (\ell + \frac{C_{\mathfrak{g}}}{2}))^{-1}\Omega_{23}\Phi_3\Phi_2$

converges in $\mathrm{Ker}(\Omega_{12}-\delta_U-(\ell_0+\frac{C_\mathfrak{g}}{2}))^\perp$ to the unique solution of $(\Omega_{12}-\delta_U-(\ell_0+\frac{C_\mathfrak{g}}{2}))x=\Omega_{23}\Phi_3\Phi_2$ and therefore to $a_1(\ell_0)$.

We claim now that $F_\ell(z)$ is holomorphic on $\mathcal{D}\times\{z|\,|z|<1\}$. Let $\Lambda\subset\mathcal{D}$ be a compact set and $M=\sup_{\ell\in\Lambda}\|a_1(\ell)\|+\|\Phi_3\Phi_2\|$. Then, for $\ell\in\Lambda$ and $n\geq 2$, the norms $n\|(\Omega_{12}-\delta_U-n(\ell+\frac{C_\mathfrak{g}}{2}))^{-1}\Omega_{23}\|$ are uniformly bounded by some constant C and therefore, by (3.2.6), the quantities $\sup_{\ell\in\Lambda}\|a_n(\ell)\|$, $n\geq 1$ are bounded by the solution of the recurrence relation

$$\alpha_1=M \tag{3.2.8}$$

$$n\alpha_n=C\sum_{m=1}^{n-1}a_m \tag{3.2.9}$$

By uniqueness, these are the coefficients of the power series expansion of $M(1-z)^{-C}$ and it follows that $F_\ell(z)=\sum_0 a_n(\ell)z^n$ is absolutely convergent, and therefore holomorphic on $\Lambda\times\{z|\,|z|<1\}$. To conclude, notice that the value of $F_\ell(z)$ on $|z|\geq 1$ are given by parallel transport from those in $|z|<1$ with respect to the connection (3.2.4). Since the latter depends analytically on ℓ we deduce that $F_\ell(z)$ is holomorphic on $\mathcal{D}\times(\mathbb{C}\setminus[1,\infty))$ as claimed $\diamond$

Remark. The above proposition should hold without any assumptions on the eigenvalues of Ω_{12} and would then imply that braiding coefficients may always be computed for generic values of the level. At present, we can only prove this for $G=\mathrm{SU}_2$.

3.3. The reduction of the KZ equation to the DF equation.

Proposition 3.3.1. *Let $\kappa\in\mathbb{C}^*$ and consider the Knizhnik–Zamolodchikov equation with values in $W=(V_{k\theta_1}\otimes V^*_{k\theta_1}\otimes V_{\theta_1}\otimes V^*_{\theta_1})^{\mathrm{SO}_{2n}}$ given by*

$$\frac{df}{dz}=\frac{1}{\kappa}\Big(\frac{\Omega_{12}-\delta_0}{z}+\frac{\Omega_{23}}{z-1}\Big)f \tag{3.3.1}$$

where δ_0 and the matrices Ω_{ij} are given by proposition 2.2.1. If e_0 is the eigenvector of Ω_{12} corresponding to the eigenvalue δ_0, the map $f\to R(f)=(f,e_0)(z-1)^{-k/\kappa}$ is an isomorphism of the space of solutions of (3.3.1) *onto the solutions of the Dotsenko–Fateev equation* (1.1) *with parameters*

$$a=\frac{2(n-1)+k}{\kappa}\qquad b=-\frac{\kappa+1}{\kappa}\qquad c=-\frac{k}{\kappa}\qquad g=-2\frac{n-2}{\kappa} \tag{3.3.2}$$

Proof. We shall use the basis of W where the matrices Ω_{ij} have the form given by proposition 2.2.1 so that $e_0=\begin{pmatrix}0\\0\\1\end{pmatrix}$. Let f be a solution of (3.3.1). By setting $g=(z-1)^{-k/\kappa}f$ and rescaling the entries of g, the equations may be written as

$$\begin{aligned}\kappa\frac{dg}{dz}=&\begin{pmatrix}2n&0&0\\0&2(n-1)&0\\0&0&0\end{pmatrix}\frac{g}{z}\\&+\begin{pmatrix}-n-k&\frac{(n-1)}{n}(k^2+2k(n-1)+n(n-2))&0\\1&-n-k+1&\frac{k}{n}(k+2(n-1))\\0&1&-k\end{pmatrix}\frac{g}{z-1}\end{aligned} \tag{3.3.3}$$

Let $g=\begin{pmatrix}u\\v\\w\end{pmatrix}$. Using the last component of (3.3.3) to express v in terms of w and the second to express u in terms of w, one finds

$$u=\frac{k(n-1)}{z}(2-\frac{n-k+2}{n}z)w+\frac{\kappa(z-1)}{z}(2(n-1)+(\kappa-n+2k+1)z)w'+\kappa^2(z-1)^2w'' \tag{3.3.4}$$

$$v=kw+\kappa(z-1)w' \tag{3.3.5}$$

Eliminating u and v from the first component of (3.3.3) yields a third order ODE for $w = R(f)$ of the form (1.1) with

$$K_1 = \frac{2n + 3(k+\kappa) - 1}{\kappa} \qquad K_2 = -2\frac{2n-1}{\kappa} \tag{3.3.6}$$

$$L_1 = \frac{(1+k+\kappa)(\kappa + 2(n+k-1))}{\kappa^2} \qquad L_2 = 2\frac{(n-1)(2n+\kappa)}{\kappa^2} \tag{3.3.7}$$

$$L_3 = -2\frac{2n^2 + 4kn + 3n\kappa - 2(n+k+\kappa)}{\kappa^2} \tag{3.3.8}$$

$$M_1 = -2\frac{k(n-1)(\kappa + 2(n+k-1))}{\kappa^3} \qquad M_2 = 2\frac{k(n-1)(2n+\kappa)}{\kappa^3} \tag{3.3.9}$$

which is indeed the Dotsenko–Fateev equation with parameters given by (3.3.2). Since $w = 0$ implies $u = v = 0$ by (3.3.4)–(3.3.5), R is injective and therefore surjective $\diamond$

The following elementary lemma justifies our use of complex values of κ or equivalently of $\ell = \kappa - 2(n-1)$. Indeed, for $\kappa \in \mathbb{R}_+$, $b = -(\kappa+1)\kappa^{-1} \in (-\infty, -1)$ and the parameters of the Dotsenko–Fateev equation given by (3.3.2) lie outside the range (1.1.4)–(1.1.5) where we have a good description of its solutions.

LEMMA 3.3.2. *The parameters a, b, c, g given by* (3.3.2) *lie in the range*

$$\Re a, \Re b, \Re c, \Re g > -1 \qquad \Re(2a+g), \Re(2b+g), \Re(2c+g) > -2 \tag{3.3.10}$$

$$\Re(a+b+c+g) < -1 \qquad \Re(2a+2b+2c+g) < -2 \tag{3.3.11}$$

if, and only if

$$\Re\kappa < 0 \qquad \text{and} \qquad |\kappa + \frac{1}{2}(2(n-1)+k)|^2 > \frac{1}{4}(2(n-1)+k)^2 \tag{3.3.12}$$

The next lemma describes the monodromic behaviour at 0 or ∞ of the scalar reduction $R(\cdot)$ of specific solutions of the Knizhnik–Zamolodchikov equations (3.3.1). When κ satisfies (3.3.12), this will allow us to identify them with the solutions of the Dotsenko–Fateev equation given in proposition 1.3.1.

LEMMA 3.3.3. *Fix a basis of eigenvectors of each of the matrices Ω_{12} and Ω_{13} given by proposition 2.2.1. Denote the solutions of* (3.3.1) *at 0 and ∞ with initial terms equal to these eigenvectors by $f_{2\theta_1}, f_{\theta_1+\theta_2}, f_0$ and $g_{(k+1)\theta_1}, g_{k\theta_1+\theta_2}, g_{(k-1)\theta_1}$ respectively where the labels refer to those of the corresponding eigenvalues. Then, up to multiplicative constants independent of κ*

$$R(g_{(k+1)\theta_1}) = \left(\frac{1}{z}\right)^{2k/\kappa}(1 + z^{-1}O(z^{-1})) \tag{3.3.13}$$

$$R(f_0) = e^{-i\pi k/\kappa}(1 + zO(z)) \qquad R(g_{k\theta_1+\theta_2}) = \left(\frac{1}{z}\right)^{(k-1)/\kappa}(1 + z^{-1}O(z^{-1})) \tag{3.3.14}$$

$$R(g_{(k-1)\theta_1}) = \left(\frac{1}{z}\right)^{-2(n-1)/\kappa}(1 + z^{-1}O(z^{-1})) \tag{3.3.15}$$

PROOF. We use the basis of $W = (V_{k\theta_1} \otimes V_{k\theta_1}^* \otimes V_{\theta_1} \otimes V_{\theta_1}^*)^{\mathrm{SO}_{2n}}$ given by proposition 2.2.1 so that the eigenvector e_0 of Ω_{12} corresponding to the eigenvalue δ_0 is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Since the Taylor series of f_0 at 0 is $e_0 + \sum_{n\geq 1} e_n z^n$, the lemma follows for $R(f_0)$. The Taylor series of g_μ at $z = \infty$ is $\left(\frac{1}{z}\right)^{\widetilde{\delta}_\mu} \sum_{n\geq 0} \xi_{n,\mu} z^{-n}$ where the $\widetilde{\delta}_\mu$ and $\xi_{0,\mu}$ are the eigenvalues and corresponding eigenvectors of the residue matrix of (3.3.1) at infinity. By (VII.1.10), the latter is $-(\Omega_{12} - \delta_0 + \Omega_{23}) = \Omega_{13}$ and the $\widetilde{\delta}_\mu$ are therefore given by proposition 2.2.1. Rescaling the basis of W so as to have

$$\Omega_{13} = \begin{pmatrix} -n & \frac{(n-1)}{n}(k^2 + 2k(n-1) + n(n-2)) & 0 \\ 1 & -n+1 & \frac{k}{n}(k + 2(n-1)) \\ 0 & 1 & 0 \end{pmatrix} \tag{3.3.16}$$

the corresponding eigenvectors are

$$v_{(k+1)\theta_1} = \begin{pmatrix} k(n-1)(k+n-2) \\ -kn \\ n \end{pmatrix} \qquad v_{k\theta_1+\theta_2} = \begin{pmatrix} k^2+2k(n-1)+n(n-2) \\ -n \\ -n \end{pmatrix} \tag{3.3.17}$$

$$v_{(k-1)\theta_1} = \begin{pmatrix} (n-1)(n+k)(k+2(n-1)) \\ n(k+2(n-1)) \\ n \end{pmatrix} \tag{3.3.18}$$

Thus, up to a multiplicative constant

$$R(g_\mu) = (z-1)^{-k/\kappa}\Big(\frac{1}{z}\Big)^{\widetilde{\delta}_\mu}(1+z^{-1}O(z^{-1})) = \Big(\frac{1}{z}\Big)^{\widetilde{\delta}_\mu+k/\kappa}(1+z^{-1}O(z^{-1})) \tag{3.3.19}$$

as claimed $\diamond$

3.4. Non–vanishing of the braiding coefficients.

THEOREM 3.4.1. *At any level $\ell_0 \in \mathbb{N}$, the following braiding identities hold*

$$\phi^{k\theta_1}_{k\theta_1\ 0}(w)\phi^{\theta_1}_{0\ \theta_1}(z) = \sum_{\mu\in\{(k-1)\theta_1, k\theta_1+\theta_2, (k+1)\theta_1\}} \beta_{k\theta_1,\mu}\phi^{\theta_1}_{k\theta_1\ \mu}(z)\phi^{k\theta_1}_{\mu\ \theta_1}(w) \tag{3.4.1}$$

if $k = 1\ldots\ell_0-1$ and

$$\phi^{\ell_0\theta_1}_{\ell_0\theta_1\ 0}(w)\phi^{\theta_1}_{0\ \theta_1}(z) = \beta_{\ell_0\theta_1,(\ell_0-1)\theta_1}\phi^{\theta_1}_{\ell_0\theta_1\ (\ell_0-1)\theta_1}(z)\phi^{\ell_0\theta_1}_{(\ell_0-1)\theta_1\ \theta_1}(w) \tag{3.4.2}$$

where the braiding coefficients $\beta_{\lambda,\mu}$ are non–zero.

PROOF. By the tensor product rule (2.2.3) and theorem VII.2.1, (3.4.1) holds for $k = 1\ldots\ell_0-1$ for some braiding coefficients $\beta_{\lambda,\mu}$. When $k = \ell_0$, the sum on the right hand–side of (3.4.1) is restricted to $\mu = (k-1)\theta_1 = (\ell_0-1)\theta_1$ since the irreducible representations of SO_{2n} with highest weight given by the other values of μ are not admissible at level ℓ_0. Thus, (3.4.2) holds for some $\beta_{\ell_0\theta_1,(\ell_0-1)\theta_1}$ whose vanishing would imply that of the four–point function of the product $\phi^{k\theta_1}_{k\theta_1\ 0}(w)\phi^{\theta_1}_{0\ \theta_1}(z)$, a contradiction. This settles (3.4.2). Let now f_0 be the reduced four–point function corresponding to the left hand–side of (3.4.1) so that f_0 satisfies (3.3.1) and the leading term of its Taylor expansion at 0 is the Ω_{12} eigenvector e_0 corresponding to the eigenvalue δ_0. Let g_μ be the four–point functions corresponding to the right hand–sides of (3.4.1) so that the initial terms of their expansion at ∞ are the eigenvectors of Ω_{13}. By corollary 2.2.2, Ω_{12} and Ω_{13} satisfy the assumptions of proposition 3.2.2 and we may therefore analytically continue f_0 and the g_μ through solutions of (3.3.1) for any $\kappa = \ell + 2(n-1)$ where $\ell \in \mathcal{D} = (\mathbb{C}\setminus\mathbb{R})\cup D$ and D is a disc centred at ℓ_0. We shall abusively denote the continuation of the four–point functions by the same symbol.

Take $\kappa = \ell + 2(n-1)$, $\ell \in \mathcal{D}$ in the range (3.3.12) such that $\operatorname{Im}\kappa \neq 0$. Then, the characteristic exponents of the Dotsenko–Fateev equation given by the monodromic behaviour of the solutions of proposition 1.3.1 do not differ by integers when the parameters a, b, c, g are bound by (3.3.2) and it follows that the solutions of the Dotsenko–Fateev equation are, up to a constant factor uniquely determined by their monodromy about 0 or ∞. Thus, using (3.3.2) to compare the exponents of $R(f_0)$ and $R(g_\mu)$ given by lemma 3.3.3 with those of the solutions $I_{z_i,j}$ given by proposition 1.3.1, we find

$$R(g_{(k+1)\theta_1}) = \rho^{-1}_{\infty,1}I_{\infty,1} \tag{3.4.3}$$

$$R(f_0) = e^{-i\pi k/\kappa}\rho^{-1}_{0,1}I_{0,1} \qquad\qquad R(g_{k\theta_1+\theta_2}) = \rho^{-1}_{\infty,2}I_{\infty,2} \tag{3.4.4}$$

$$R(g_{(k-1)\theta_1}) = \rho^{-1}_{\infty,3}I_{\infty,3} \tag{3.4.5}$$

and therefore, by proposition 1.4.1

$$\begin{aligned}R(f_0) = {} & e^{-i\pi k/\kappa}\rho_{0,1}^{-1}\rho_{\infty,1}\frac{\mathrm{s}(a)\,\mathrm{s}(a+\frac{g}{2})}{\mathrm{s}(a+b+\frac{g}{2})\,\mathrm{s}(a+b+g)}R(g_{(k+1)\theta_1}) \\ & + e^{-i\pi k/\kappa}\rho_{0,1}^{-1}\rho_{\infty,2}2e^{-\pi i(a+c+\frac{g}{2})}\,\mathrm{c}(\frac{g}{2})\frac{\mathrm{s}(a)\,\mathrm{s}(c)}{\mathrm{s}(a+b)\,\mathrm{s}(a+b+g)}R(g_{k\theta_1+\theta_2}) \\ & + e^{-i\pi k/\kappa}\rho_{0,1}^{-1}\rho_{\infty,3}\frac{\mathrm{s}(c)\,\mathrm{s}(c+\frac{g}{2})}{\mathrm{s}(a+b+\frac{g}{2})\,\mathrm{s}(a+b+\frac{g}{2})}R(g_{(k-1)\theta_1})\end{aligned} \tag{3.4.6}$$

where a, b, c, g are given by (3.3.2) and the $\rho_{z_i,j}$ by proposition 1.3.1. Since R is injective the same linear relation binds f_0 and the g_μ. Continuing back to $\kappa = \ell_0 + 2(n-1)$ therefore yields

$$f_0 = \beta_{(k+1)\theta_1}g_{(k+1)\theta_1} + \beta_{k\theta_1+\theta_2}g_{k\theta_1+\theta_2} + \beta_{(k-1)\theta_1}g_{(k-1)\theta_1} \tag{3.4.7}$$

where the coefficients β_μ are obtained by evaluating those of (3.4.6) and, by direct inspection, do not vanish if $1 \leq k \leq \ell_0 - 1$ $\diamond$

Remark. The coefficients involved in the braiding of the primary fields with charge the vector representation of SO_{2n} and its second exterior power $V_{\theta_1+\theta_2}$ may in principle be obtained by studying the Knizhnik–Zamolodchikov equations with values in $(V_{\theta_1+\theta_2}\otimes V^*_{\theta_1+\theta_2}\otimes V_{\theta_1}\otimes V^*_{\theta_1})^{\mathrm{SO}_{2n}}$. A computation similar to the proof of proposition 2.2.1 shows in fact that the corresponding operators Ω_{ij} are

$$\Omega_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2n+1 \end{pmatrix} \qquad \Omega_{23} = \begin{pmatrix} -n & x & 0 \\ \widetilde{x} & -n+1 & y \\ 0 & \widetilde{y} & 0 \end{pmatrix} \tag{3.4.8}$$

where

$$x\widetilde{x} = \frac{(n+1)(n-2)^2}{n} \qquad \text{and} \qquad y\widetilde{y} = 4\frac{n-1}{n} \tag{3.4.9}$$

In spite of the similarity of (3.4.8) and (2.2.6), the corresponding Knizhnik–Zamolodchikov equations, when reduced to a third order scalar equation are not of the Dotsenko–Fateev form so that no explicit solutions are readily available.

Remark. The computations of this chapter apply, with very little change to the group $G = \mathrm{SO}_{2n+1}$ and may be used to show that the braiding relations analogous to those of theorem 3.4.1 only involve non–zero coefficients.

Remark. The solutions (1.6) were discovered by Dotsenko and Fateev in the context of minimal models [**DF**]. They were subsequently generalised by Schechtman–Varchenko [**SV**] and Feigin–Frenkel [**EdF**] who used them systematically to give solutions of all KZ equations. It is interesting to note however that their generalised hypergeometric solutions, which first appeared in the work of Aomoto and Gelfand [**Ao, GKZ**], use Euler–like contour integrals with a number of integration variables growing linearly in n, even in the specific cases considered in this chapter. This makes them intractable for computational purposes.

Part 3

Fusion of positive energy representations

CHAPTER IX

Connes fusion of positive energy representations of $L\,\mathrm{Spin}_{2n}$

In this final chapter, we define a tensor product operation $\boxtimes$ or *Connes fusion* on the category $\mathcal{P}_\ell$ of positive energy representations of $L\,\mathrm{Spin}_{2n}$ at a fixed level ℓ and study the resulting algebraic structure on $\mathcal{P}_\ell$.

In section 1 we give the definition and elementary properties of Connes fusion. In section 2, we use the action of the centre $Z(\mathrm{Spin}_{2n})$ on $\mathcal{P}_\ell$ via conjugation by discontinuous loops to compute the fusion with representations lying on the orbit of the vacuum $\mathcal{H}_0$. As a simple corollary, we show that the level 1 fusion ring of $L\,\mathrm{Spin}_{2n}$ is isomorphic to the group algebra of $Z(\mathrm{Spin}_{2n})$. In section 3, we derive braiding identities for bounded intertwiners for the local loop groups $L_I\,\mathrm{Spin}_{2n}$ from the ones satisfied by smeared primary fields. These are used in section 4 to give an upper bound for the fusion with the vector representation $\mathcal{H}_\Box$ in terms of the *Verlinde rules*. The rest of the chapter is devoted to showing that this bound is actually attained. In section 5, we prove this for the fusion of $\mathcal{H}_\Box$ with its symmetric powers by using the braiding computations of chapter VIII. Section 6 initiates the study of the ring $\mathcal{R}_0$ generated by the irreducible summands of the iterated fusion products of $\mathcal{H}_\Box$. We prove that $\mathcal{R}_0$ is commutative by using a *braiding operator* B which gives an isomorphism $\mathcal{H}_1\boxtimes\mathcal{H}_2 \cong \mathcal{H}_2\boxtimes\mathcal{H}_1$ for any $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{R}_0$. In section 7 we compute the eigenvalues of the braiding operator acting on the fusion of $\mathcal{H}_\Box$ with itself. Section 8 describes some important algebraic properties of $\mathcal{R}_0$ – most notably the existence of a *quantum dimension function* compatible with fusion – which are derived from the Doplicher–Haag–Roberts theory of superselection sectors. In section 9, we compute the quantum dimension of $\mathcal{H}_\Box$ by using some arguments of Wenzl and the fusion rules obtained in section 5. Section 10 gives an alternative computation of this dimension under the assumption that the Verlinde rules hold. Reassuringly, we find the value obtained in section 9. Section 11 contains the main results of this thesis. By using a Perron–Frobenius argument based on the coincidence of the two computations of the quantum dimension of $\mathcal{H}_\Box$, we show that its fusion with the positive energy representations of $L\,\mathrm{Spin}_{2n}$ whose lowest energy subspace is a single–valued SO_{2n}–module is given by the Verlinde rules. This implies in particular that these representations are closed under fusion and form a commutative and associative ring. Finally, using the action of discontinuous loops on $\mathcal{P}_\ell$, we extend the previous results to all positive energy representations of $L\,\mathrm{Spin}_{2n}$ when the level is odd.

1. Definition of Connes fusion

This section follows [**Wa3**, §30] and rests on the results describing the von Neumann algebras generated by local loop groups in positive energy representations obtained in chapter IV. Fix $\ell \in \mathbb{N}$ and consider the set $\mathcal{P}_\ell$ of positive energy representations of LG at level ℓ. Here and in what follows, $G = \mathrm{Spin}_{2n}$, $n \geq 3$.

Let $(\pi_0, \mathcal{H}_0) \in \mathcal{P}_\ell$ be the vacuum representation and denote by $\mathcal{L}G = \pi_0^* U(\mathcal{H}_0)$ and $\mathcal{L}_I G$ the corresponding central extension of LG and its restriction to a local loop group $L_I G$. By proposition I.2.4.3, $\mathcal{L}G$ is isomorphic to $\pi_i^* U(\mathcal{H}_i)$ for any $(\pi_i, \mathcal{H}_i) \in \mathcal{P}_\ell$ and therefore acts unitarily on $\mathcal{H}_i$. We denote the corresponding representation by the same symbol. The restriction of π_i to $\mathcal{L}_I G$ has the simple form $\pi_i(\gamma) = U\pi_0(\gamma)U^*$ where $U : \mathcal{H}_0 \to \mathcal{H}_i$ is a unitary equivalence of $L_I G$–modules provided by local equivalence. Indeed, both π_i and conjugation by U yield isomorphisms $\mathcal{L}_I G \cong \pi_i^* U(\mathcal{H}_i)\big|_{L_I G}$ and therefore differ by a character of $L_I G$ which, by lemma IV.1.1.1 is necessarily trivial. It follows that the restriction of π_i to $L_I G$ extends to the canonical spatial isomorphism $\pi_0(L_I G)'' \cong \pi_i(L_I G)''$.

In particular, if $x \in \mathrm{Hom}_{\mathcal{L}_I G}(\mathcal{H}_0, \mathcal{H}_i)$, then $xa = \pi_i(a)x$ for any $a \in \pi_0(L_I G)''$.

Let now $\mathcal{H}_i, \mathcal{H}_j \in \mathcal{P}_\ell$. By locality, we may regard each as a bimodule over the pair $(\mathcal{L}_I G, \mathcal{L}_{I^c} G)$ and form the intertwiner spaces

$$\mathfrak{X}_i = \mathrm{Hom}_{\mathcal{L}_{I^c} G}(\mathcal{H}_0, \mathcal{H}_i) \qquad \text{and} \qquad \mathfrak{Y}_j = \mathrm{Hom}_{\mathcal{L}_I G}(\mathcal{H}_0, \mathcal{H}_j) \tag{1.1}$$

where $\mathcal{H}_0$ is the vacuum representation at level ℓ with highest weight vector Ω. These are $\mathcal{L}_I G$ and $\mathcal{L}_{I^c} G$–modules respectively.

LEMMA 1.1. *The maps $\mathfrak{X}_i \to \mathcal{H}_i$ and $\mathfrak{Y}_j \to \mathcal{H}_j$ given by $x \to x\Omega$ and $y \to y\Omega$ are embeddings with dense image.*

PROOF. If $x \in \mathfrak{X}_i$ and $x\Omega = 0$, then $x\pi_0(\mathcal{L}_{I^c} G)''\Omega = \pi_i(\mathcal{L}_{I^c} G)'' x\Omega = 0$ and therefore, by the Reeh–Schlieder theorem, $x = 0$. To prove the density of the embedding, we may assume that $\mathcal{H}_i$ is irreducible. Pick a unitary $u_i \in \mathrm{Hom}_{\mathcal{L}_{I^c} G}(\mathcal{H}_0, \mathcal{H}_i)$ using local equivalence. Then, by Haag duality $\mathfrak{X}_i = u_i \pi_0(\mathcal{L}_{I^c} G)' = u_i \pi_0(\mathcal{L}_I G)''$ and therefore, by the Reeh–Schlieder theorem $\overline{\mathfrak{X}_i \Omega} = u_i \overline{\pi_0(\mathcal{L}_I G)''\Omega} = \mathcal{H}_i$. The fact that $\mathfrak{Y}_j$ embeds densely in $\mathcal{H}_j$ is proved identically $\diamond$

Consider the hermitian form $\langle \cdot, \cdot \rangle$ on the algebraic tensor product $\mathfrak{X}_i \otimes \mathfrak{Y}_j$ defined by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = (x_2^* x_1 y_2^* y_1 \Omega, \Omega) \tag{1.2}$$

where the inner product is taken in $\mathcal{H}_0$.

LEMMA 1.2. *The form $\langle \cdot, \cdot \rangle$ is positive semi–definite.*

PROOF. Let $z = \sum_{p=1}^n x_p \otimes y_p \in \mathfrak{X}_i \otimes \mathfrak{Y}_j$ so that $\langle z, z \rangle = \sum_{p,q} (x_q^* x_p y_q^* y_p \Omega, \Omega)$. Let $\{e_p\}_{p=1}^n$ be an orthonormal basis of $\mathbb{C}^n$ and set $\Omega_p = \Omega \otimes e_p \in \mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^n$. Consider the non–negative $n \times n$ matrices X, Y acting on $\mathcal{H}$ with entries $x_p^* x_q \in \pi_0(\mathcal{L}_{I^c} G)'$ and $y_q^* y_p \in \pi_0(\mathcal{L}_I G)'$ respectively. By Haag duality, X and Y commute and therefore $\langle z, z \rangle = \sum_p (XY\Omega_p, \Omega_p) \geq 0$ $\diamond$

DEFINITION. The *Connes fusion* $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is the Hilbert space completion of the quotient of $\mathfrak{X}_i \otimes \mathfrak{Y}_j$ by the radical of $\langle \cdot, \cdot \rangle$. $\mathcal{H}_i \boxtimes \mathcal{H}_j$ supports commuting unitary actions of $\mathcal{L}_I G \times \mathcal{L}_{I^c} G$ given by $(\gamma_I, \gamma_{I^c}) x \otimes y = \pi_i(\gamma_I) x \otimes \pi_j(\gamma_{I^c}) y$ which are easily seen to be strongly continuous.

REMARK. The Connes fusion $\mathcal{H}_i \boxtimes \mathcal{H}_j$ of two irreducibles $\mathcal{H}_i, \mathcal{H}_j \in \mathcal{P}_\ell$ is *a priori* only an $(\mathcal{L}_I G, \mathcal{L}_{I^c} G)$–bimodule since the action of $(\mathcal{L}_I G, \mathcal{L}_{I^c} G)$ need not necessarily extend to one of $\mathcal{L}G$, let alone a positive energy one of $\mathcal{L}G \rtimes \mathrm{Rot}\, S^1$. One of our main results (theorems 11.3 and 11.5) states that this action does extend, and therefore that $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is a positive energy representation of LG, when ℓ is odd or the lowest energy subspaces of $\mathcal{H}_i, \mathcal{H}_j$ are single–valued SO_{2n}–modules.

LEMMA 1.3. $\mathcal{H}_0 \boxtimes \mathcal{H}_i \cong \mathcal{H}_i \cong \mathcal{H}_i \boxtimes \mathcal{H}_0$

PROOF. The map $U : \mathfrak{X}_0 \otimes \mathfrak{Y}_i \to \mathcal{H}_i$, $x \otimes y \to yx\Omega$ is readily seen to be an $\mathcal{L}_I G \times \mathcal{L}_{I^c} G$–equivariant isometry. By lemma 1.1, $\overline{U(\mathfrak{X}_0 \otimes \mathfrak{Y}_i)} = \overline{\mathfrak{Y}_i \mathfrak{X}_0 \Omega} = \overline{\mathfrak{Y}_i \mathcal{H}_0} = \mathcal{H}_i$ and U therefore extends to a unitary $\mathcal{H}_0 \boxtimes \mathcal{H}_i \to \mathcal{H}_i$. Similarly, the unitary equivalence $\mathcal{H}_i \boxtimes \mathcal{H}_0 \cong \mathcal{H}_i$ is given by $y \otimes x \to yx\Omega$ $\diamond$

PROPOSITION 1.4. *Connes fusion is associative, that is $\mathcal{H}_i \boxtimes (\mathcal{H}_j \boxtimes \mathcal{H}_k) \cong (\mathcal{H}_i \boxtimes \mathcal{H}_j) \boxtimes \mathcal{H}_k$.*

PROOF. We follow [**Lo**]. Notice that for any $x_p \in \mathfrak{X}_i$ and $y_q \in \mathfrak{Y}_j$,

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = (\pi_i(y_2^* y_1) x_1 \Omega, x_2 \Omega) = (\pi_j(x_2^* x_1) y_1 \Omega, y_2 \Omega) \tag{1.3}$$

where, by Haag duality $y_2^* y_1 \in \pi_0(\mathcal{L}_I G)' = \pi_0(\mathcal{L}_{I^c} G)''$ and $x_2^* x_1 \in \pi_0(\mathcal{L}_I G)''$ so that, by local equivalence, they may be represented on $\mathcal{H}_i$ and $\mathcal{H}_j$ respectively. It follows from lemma 1.1 that $\mathcal{H}_i \boxtimes \mathcal{H}_j$ may equivalently be defined as the Hilbert space completion of $\mathcal{H}_i \otimes \mathfrak{Y}_j$ or $\mathfrak{X}_i \otimes \mathcal{H}_j$ with respect to the forms

$$(\xi_1 \otimes y_1, \xi_2 \otimes y_2) = (\pi_i(y_2^* y_1)\xi_1, \xi_2) \qquad \text{and} \qquad (x_1 \otimes \eta_1, x_2 \otimes \eta_2) = (\pi_j(x_2^* x_1)\eta_1, \eta_2) \tag{1.4}$$

respectively. More generally, $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is the completion with respect to these forms of $\mathcal{K}_i \otimes \mathfrak{Y}_j$ or $\mathfrak{X}_i \otimes \mathcal{K}_j$, where the $\mathcal{K}_p \subset \mathcal{H}_p$ are dense subspaces. With these alternative descriptions of Connes

fusion at hand, a unitary giving the claimed isomorphism may be densely defined as the $\mathcal{L}_I G\times\mathcal{L}_{I^c}G$–equivariant map

$$\mathfrak{X}_i\otimes(\mathcal{H}_j\otimes\mathfrak{Y}_k)\longrightarrow(\mathfrak{X}_i\otimes\mathcal{H}_j)\otimes\mathfrak{Y}_k,\ x\otimes(\eta\otimes z)\longrightarrow(x\otimes\eta)\otimes z \tag{1.5}$$

$\diamond$

REMARK. As stated, proposition 1.4 requires $\mathcal{H}_j\boxtimes\mathcal{H}_k$ and $\mathcal{H}_i\boxtimes\mathcal{H}_j$ to be of positive energy and we shall use it in those cases only. We note however that Connes fusion may be defined in the larger category of bimodules over the III_1 factors $(\pi_0(L_IG)'',\pi_0(L_{I^c}G)'')$ where the proof of proposition 1.4 shows that it is an associative operation [**Wa2**].

2. Fusion with quasi–vacuum representations and the level 1 fusion ring

Recall from section I.3 that the centre of G acts on the positive energy representations at level ℓ via conjugation by discontinuous loops. If $\zeta\in L_ZG$ and $(\pi,\mathcal{H})\in\mathcal{P}_\ell$, we denote the corresponding conjugated representation by $(\zeta_*\pi,\zeta\mathcal{H})$. We call the $\zeta\mathcal{H}_0$ *quasi–vacuum* representations in view of the following

PROPOSITION 2.1. *Let $(\zeta_*\pi_0,\zeta\mathcal{H}_0)$ be a quasi–vacuum representation. Then, Haag duality holds*

$$\zeta_*\pi_0(\mathcal{L}_IG)'=\zeta_*\pi_0(\mathcal{L}_{I^c}G)'' \tag{2.1}$$

PROOF. This follows from Haag duality for the vacuum representation and the fact that conjugation by ζ normalises the local loop groups $\diamond$

PROPOSITION 2.2. *Let $\zeta\mathcal{H}_0$ be a quasi–vacuum representation, then $\zeta\mathcal{H}_0\boxtimes\mathcal{H}_i\cong\zeta\mathcal{H}_i\cong\mathcal{H}_i\boxtimes\zeta\mathcal{H}_0$.*

PROOF. By multiplying ζ by a suitable element in LG, we may assume that it is equal to 1 on I^c. Then, $\zeta\mathfrak{X}_0=\mathrm{Hom}_{\mathcal{L}_{I^c}G}(\mathcal{H}_0,\zeta\mathcal{H}_0)=\pi_0(\mathcal{L}_{I^c}G)'=\pi_0(\mathcal{L}_IG)''$. The map $\zeta\mathfrak{X}_0\otimes\mathfrak{Y}_i\rightarrow\mathcal{H}_i$, $x\otimes y\rightarrow yx\Omega$ is then easily seen to extend to a unitary $\zeta\mathcal{H}_0\boxtimes\mathcal{H}_i\rightarrow\zeta\mathcal{H}_i$ intertwining $\mathcal{L}_IG\times\mathcal{L}_{I^c}G$. The isomorphism $\mathcal{H}_i\boxtimes\zeta\mathcal{H}_0\cong\zeta\mathcal{H}_i$ follows in a similar way by choosing ζ supported in I^c $\diamond$

As a simple application of the above, we determine the level 1 fusion ring of $L\,\mathrm{Spin}_{2n}$.

THEOREM 2.3. *The level 1 representations of $L\,\mathrm{Spin}_{2n}$ are closed under fusion. Moreover, if $\mathcal{H}_0$ is the vacuum representation, then*

$$z\longrightarrow z\mathcal{H}_0 \tag{2.2}$$

yields an isomorphism of the group algebra $\mathbb{C}[Z(\mathrm{Spin}_{2n})]$ and the level 1 fusion ring of $L\,\mathrm{Spin}_{2n}$. Explicitly, if $\mathcal{H}_0,\mathcal{H}_v,\mathcal{H}_{s_\pm}$ are the irreducible level 1 representations, then

$$\mathcal{H}_v\boxtimes\mathcal{H}_v\cong\mathcal{H}_0 \qquad\qquad \mathcal{H}_v\boxtimes\mathcal{H}_{s_\pm}\cong\mathcal{H}_{s_\mp} \tag{2.3}$$

$$\mathcal{H}_{s_\pm}\boxtimes\mathcal{H}_{s_\pm}\cong\begin{cases}\mathcal{H}_0 & \text{if } n \text{ even}\\ \mathcal{H}_v & \text{if } n \text{ odd}\end{cases} \qquad\qquad \mathcal{H}_{s_\pm}\boxtimes\mathcal{H}_{s_\mp}\cong\begin{cases}\mathcal{H}_v & \text{if } n \text{ even}\\ \mathcal{H}_0 & \text{if } n \text{ odd}\end{cases} \tag{2.4}$$

PROOF. The map (2.2) is bijective since, by corollary I.3.2.5, $Z(\mathrm{Spin}_{2n})$ acts freely and transitively on the irreducible level 1 representations. Moreover, by proposition 2.2,

$$z_1\mathcal{H}_0\boxtimes z_2\mathcal{H}_0\cong z_1(z_2\mathcal{H}_0)\cong z_1z_2\mathcal{H}_0 \tag{2.5}$$

so that (2.2) is a ring isomorphism. Let $\mathcal{H}_\lambda$ be the irreducible, level 1 representation with highest weight λ. As noted in §I.2.3, λ is a minimal dominant weight since Spin_{2n} is simply–laced and therefore, by theorem I.3.2.3 and the explicit action of $Z(\mathrm{Spin}_{2n})$ on the level 1 alcove given by proposition I.3.1.5, $\mathcal{H}_\lambda=\exp_T(-2\pi i\lambda)\mathcal{H}_0$. Consequently, $\mathcal{H}_\lambda\boxtimes\mathcal{H}_\mu=\mathcal{H}_\nu$ where ν is the unique minimal dominant weight in the root lattice coset of $\lambda+\mu$. The fusion rules (2.3)–(2.4) now follow from the tables in §3.3 of chapter I $\diamond$

3. Braiding properties of bounded intertwiners

This section follows [**Wa2**]. We derive braiding relations satisfied by *bounded* intertwiners for the local loop groups $\mathcal{L}_I G$ from the corresponding ones for smeared primary fields. These will be used in the next section to give an upper bound for the fusion with the vector representation.

Using Young diagrams, denote by $V_\square$ the vector representation of SO_{2n} and by $\mathcal{H}_\square$ the corresponding positive energy representation at level ℓ. Let $\mathcal{H}_i$ be an irreducible positive energy representation with lowest energy subspace V_i and consider the braiding relation

$$\phi^i_{i\,0}(w)\phi^{\overline{\square}}_{0\,\square}(z) = \sum_j \lambda_j \phi^{\overline{\square}}_{i\,j}(z)\phi^i_{j\,\square}(w) \tag{3.1}$$

and the abelian braiding

$$\phi^i_{j\,\square}(w)\phi^{\square}_{\square\,0}(z) = \epsilon_j \phi^{\square}_{j\,i}(z)\phi^i_{i\,0}(w) \tag{3.2}$$

where j labels the irreducible summands V_j of $V_i \otimes V_\square$ which are admissible at level ℓ. Since $V_\square$ is minimal, each V_j has multiplicity one by proposition I.2.2.2. We normalise the above as follows. The initial terms of the primary fields $\phi^i_{i\,0}, \phi^{\square}_{\square\,0}, \phi^{\overline{\square}}_{0\,\square}$ with vertices $\begin{pmatrix} V_i \\ V_i\ \mathbb{C}\end{pmatrix}, \begin{pmatrix} V_\square \\ V_\square\ \mathbb{C}\end{pmatrix}, \begin{pmatrix} V_\square^* \\ \mathbb{C}\ V_\square\end{pmatrix}$ are the corresponding canonical Spin_{2n}–intertwiners. Fix for any j a generator $\varphi_j \in \mathrm{Hom}_{\mathrm{Spin}_{2n}}(V_\square \otimes V_i, V_j) \cong \mathbb{C}$. This determines initial terms for $\phi^i_{j\,\square}, \phi^{\square}_{j\,i}$ and $\phi^{\overline{\square}}_{i\,j}$ by taking $\varphi_j, \varphi_j\sigma$ and $(\varphi_j\sigma)^*$ respectively, where $\sigma : V_i \otimes V_\square \rightarrow V_\square \otimes V_i$ is permutation. With these normalisations, λ_j depends upon φ_j only up to a positive factor and, by (i) of lemma VII.3.1, ϵ_j is independent of the choice of φ_j and of modulus 1.

Let now $f \in C^\infty(S^1, V_i)$ and $g \in C^\infty(S^1, V_\square)$ be supported in I, I^c respectively. Since all the vertices above involve one of the minimal representations $\mathbb{C}, V_\square$, all primary fields in (3.1)–(3.2) extend to operator–valued distributions by theorem VI.3.1. We may therefore use proposition VII.4.1 to smear the braiding relations and find

$$\phi^i_{i\,0}(f)\phi^{\overline{\square}}_{0\,\square}(\overline{g}) = \sum_j \lambda_j \phi^{\overline{\square}}_{i\,j}(\overline{g}e_{\alpha_j})\phi^i_{j\,\square}(fe_{-\alpha_j}) \tag{3.3}$$

$$\phi^i_{j\,\square}(fe_{-\alpha_j})\phi^{\square}_{\square\,0}(g) = \epsilon_j \phi^{\square}_{j\,i}(ge_{-\alpha_j})\phi^i_{i\,0}(f) \tag{3.4}$$

where $\alpha_j = \Delta_i + \Delta_\square - \Delta_j$. With our normalisations, $\phi^{\overline{\square}}_{0\,\square} = (\phi^{\square}_{\square\,0})^*$ and therefore $\phi^{\overline{\square}}_{0\,\square}(\overline{g}) \subseteq (\phi^{\square}_{\square\,0}(g))^*$ which is in fact an equality since the vector primary field is bounded by theorem VI.3.1. Rewriting the above more synthetically, it follows that there exist operators $x_{qp}, y_{qp} : \mathcal{H}_p^\infty \rightarrow \mathcal{H}_q^\infty$ with the y_{qp} bounded such that

$$x_{i0}y^*_{\square 0} = \sum_j \lambda_j y^*_{ji} x_{j\square} \qquad\qquad x_{j\square}y_{\square 0} = \epsilon_j y_{ji}x_{i0} \tag{3.5}$$

Moreover, by proposition VI.4.1 x_{qp}, y_{qp} commute with $\mathcal{L}_{I^c}G$ and $\mathcal{L}_I G$ respectively.

Remark. For suitable choices of f and g, none of the above products of operators, and *a fortiori* their individual factors are zero. More precisely, if $h_0 = \int h(\theta)\frac{d\theta}{2\pi}$ is the constant term in the Fourier expansion of h, we henceforth choose f and g with $f_0, (fe_{-\alpha_j})_0, g_0, (ge_{-\alpha_j})_0 \neq 0$ for any j. Then, for any $v_j \in V_j = \mathcal{H}_j(0)$

$$(y_{ji}x_{i0}\Omega, v_j) = (\phi^{\square}_{ji}((ge_{-\alpha_j})_0, 0)\phi^i_{i0}(f_0, 0)\Omega, v_j) \tag{3.6}$$

The vanishing of the above for any v_j implies that of the initial term of one of the primary fields and therefore of the primary field itself, a contradiction. Thus, $\xi_j = y_{ji}x_{i0}\Omega \neq 0$. The non–zeroness of the other products follows similarly.

We show below that the operators in (3.5) may be replaced by bounded intertwiners without altering the braiding relations in such a way that x_{i0} and $y_{\square 0}$ become unitaries. We need a preliminary

Lemma 3.1. *The constants $\mu_j = \epsilon_j \lambda_j$ are non–negative and therefore $\lambda_j \epsilon_j = |\lambda_j|$.*

Proof. Consider the functional on $\mathcal{L}_I G \times \mathcal{L}_{I^c} G$ given by the four–point function with insertions formula

$$(\gamma_I, \gamma_{I^c}) \to (\pi_i(\gamma_I) x_{i0} y^*_{\square 0} \pi_\square(\gamma_{I^c}) y_{\square 0}\Omega, x_{i0}\Omega) \tag{3.7}$$

Since $y_{\square 0}$ is bounded, $y^*_{\square 0}\pi_\square(\gamma_{I^c})y_{\square 0} \in \pi_0(\mathcal{L}_I G)' = \pi_0(\mathcal{L}_{I^c}G)''$ and we may rewrite (3.7) as

$$(\pi_i(\gamma_I)\pi_i(y^*_{\square 0}\pi_\square(\gamma_{I^c})y_{\square 0})x_{i0}\Omega, x_{i0}\Omega) = (\pi_i(y^*_{\square 0}\pi_\square(\gamma_I)\pi_\square(\gamma_{I^c})y_{\square 0})x_{i0}\Omega, x_{i0}\Omega) \tag{3.8}$$

so that it extends to a positive linear functional on the algebraic tensor product $\mathcal{A} = \pi_0(\mathcal{L}_I G)'' \otimes \pi_0(\mathcal{L}_{I^c}G)''$. On the other hand, using the braiding relations (3.5) we may write (3.7) as

$$\sum_j \lambda_j (\pi_j(\gamma_I)\pi_j(\gamma_{I^c})x_{j\square}y_{\square 0}\Omega, y_{ji}x_{i0}\Omega) = \sum_j \lambda_j\epsilon_j(\pi_j(\gamma_I)\pi_j(\gamma_{I^c})y_{ji}x_{i0}\Omega, y_{ji}x_{i0}\Omega) \tag{3.9}$$

so that the functional $\phi : a_I \otimes b_{I^c} \to \sum_j \epsilon_j\lambda_j(\pi_j(a_I)\pi_j(b_{I^c})\xi_j, \xi_j)$, where $\xi_j = y_{ji}x_{i0}\Omega$, is positive on $\mathcal{A}$. By von Neumann density, the $\mathcal{H}_j$ involved are irreducible and inequivalent $\mathcal{A}$-modules, and therefore $(\oplus_j \pi_j(\mathcal{A}))'' = \bigoplus_j \mathcal{B}(\mathcal{H}_j)$. In particular, any $T \in \mathcal{B}(\mathcal{H}_j)$ is a strong limit of elements of the form $\oplus_j\pi_j(a)$ so that T^*T is a weak limit of elements $\oplus_j\pi_j(a^*a)$ and therefore $\phi(T^*T) \geq 0$. Choosing $T = 1_j$, we find $\phi(T^*T) = \lambda_j\epsilon_j\|\xi_j\|^2 \geq 0$ which, in view of the remark following (3.5), implies $\lambda_j\epsilon_j \geq 0$ $\diamond$

Lemma 3.2. *Let $\mathcal{H}_p^\infty$ be dense subspaces of the Hilbert spaces $\mathcal{H}_p$, $p = 1 \dots 4$ and consider operators*

$$\begin{tikzcd} \mathcal{H}_1^\infty \arrow[r, "T"] \arrow[d, "u"'] & \mathcal{H}_2^\infty \arrow[d, "v"] \\ \mathcal{H}_3^\infty \arrow[r, "S"'] & \mathcal{H}_4^\infty \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{H}_1^\infty \arrow[r, "T"] & \mathcal{H}_2^\infty \\ \mathcal{H}_3^\infty \arrow[r, "S"'] \arrow[u, "u^*"] & \mathcal{H}_4^\infty \arrow[u, "v^*"'] \end{tikzcd} \tag{3.10}$$

with T, S closeable and u, v bounded. If the above diagrams are commutative, they remain so when T and S are replaced by their phases.

Proof. By the boundedness of u, v, we get $v\overline{T} \subseteq \overline{S}u$ and, in particular $u\mathcal{D}(\overline{T}) \subseteq \mathcal{D}(\overline{S})$. Similarly, $v^*\overline{S} \subseteq \overline{T}u^*$ whence, taking adjoints, $uT^* \subseteq S^*v$ and $v\mathcal{D}(T^*) \subseteq \mathcal{D}(S^*)$. We claim that $uT^*\overline{T} \subseteq S^*\overline{S}u$. To see this, let $\xi \in \mathcal{D}(T^*\overline{T})$. Then $\xi \in \mathcal{D}(\overline{T})$ so that $u\xi \in \mathcal{D}(\overline{S})$ and $\overline{S}u\xi = v\overline{T}\xi \in \mathcal{D}(S^*)$ since $\overline{T}\xi \in \mathcal{D}(T^*)$. It follows that $S^*\overline{S}u\xi = S^*v\overline{T}\xi = uT^*\overline{T}\xi$ as claimed. By functional calculus, $uf(T^*\overline{T}) = f(S^*\overline{S})u$ for any bounded measurable function f. In particular,

$$v\overline{T}(T^*\overline{T}+\epsilon)^{-\frac{1}{2}} = \overline{S}u(T^*\overline{T}+\epsilon)^{-\frac{1}{2}} = \overline{S}(S^*\overline{S}+\epsilon)^{-\frac{1}{2}}u \tag{3.11}$$

which, in the limit $\epsilon \to 0$ yields the commutativity of the first diagram when T and S are replaced by their phases. That of the second follows by the above argument by permuting (T, S), (u, u^*), (v, v^*), $(\mathcal{H}_1, \mathcal{H}_3)$ and $(\mathcal{H}_2, \mathcal{H}_4)$ $\diamond$

Proposition 3.3. *There exist non–zero bounded operators $x_{qp}, y_{qp} : \mathcal{H}_p \to \mathcal{H}_q$ commuting with $\mathcal{L}_{I^c}G$ and $\mathcal{L}_I G$ respectively such that*

$$x_{i0}y^*_{\square 0} = \sum_j \lambda_j y^*_{ji}x_{j\square} \qquad\qquad x_{j\square}y_{\square 0} = \epsilon_j y_{ji}x_{i0} \tag{3.12}$$

where $x_{i0}, y_{\square 0}$ are unitary and λ_j, ϵ_j are the braiding coefficients (3.1)–(3.2).

Proof. We start from the relations (3.5) satisfied by the non–zero operators x_{qp}, y_{qp} and modify the x_{qp} in various steps without altering the y_{qp} and the braiding relations (3.5).

1st step. Applying lemma 3.2 to

$$\begin{tikzcd} \mathcal{H}_0^\infty \arrow[r, "x_{i0}"] \arrow[d, "y_{\square 0}"'] & \mathcal{H}_i^\infty \arrow[d, "\oplus_j|\lambda_j|^{\frac{1}{2}}\epsilon_j y_{ji}"] \\ \mathcal{H}_\square^\infty \arrow[r, "\oplus_j|\lambda_j|^{\frac{1}{2}}x_{j\square}"'] & \bigoplus_j \mathcal{H}_j^\infty \end{tikzcd} \tag{3.13}$$

we may replace the x_{qp} in (3.5) by bounded operators. This yields a non–zero x_{i0}, since it is simply the phase of the former x_{i0}.

2nd step. We wish to modify x_{i0} to make it injective with dense image. Another application of lemma 3.2 will then allow its replacement by a unitary phase. Although we shall modify the range $R(x_{i0})$ first, notice that x_{i0} may always be assumed to be injective. Indeed, the projection onto $\mathrm{Ker}(x_{i0})^\perp$ lies in $\pi_0(\mathcal{L}_{I^c}G)' = \pi_0(\mathcal{L}_I G)''$. Since the latter is a type III factor, there exists a partial isometry $u \in \pi_0(\mathcal{L}_I G)''$ with initial space $\mathcal{H}_0$ and final space $\mathrm{Ker}(x_{i0})^\perp$. Replacing each x_{pq} by $x_{pq}\pi_q(u)$ yields an injective x_{i0} without altering the braiding relations.

3rd step. Making x_{i0} surjective is a trifle more involved since in general $\pi_i(\mathcal{L}_{I^c}G)' \subsetneq \pi_i(\mathcal{L}_I G)''$ and the previous device does not apply. We resort to an averaging procedure relying on von Neumann density or equivalently the fact that $\pi_i(\mathcal{L}_{I^c}G)' \cap \pi_i(\mathcal{L}_I G)' = \mathbb{C}$. Let $\{g_n\}$ be a dense, countable subgroup of $U(\pi_i(\mathcal{L}_I G)'')$ and $u_m \in \pi_0(\mathcal{L}_I G)''$ partial isometries satisfying $u_m^* u_n = \delta_{mn}$. We may replace each x_{qp} by the norm convergent sum $\sum_n 2^{-n}\pi_q(g_n)x_{qp}\pi_p(u_n)$ without altering the braiding relations. If $p \in \pi_i(\mathcal{L}_{I^c}G)'$ is the projection onto $R(x_{i0}) = \mathrm{Ker}(x_{i0}x_{i0}^*)^\perp$, the corresponding projection for $\widetilde{x}_{i0} = \sum_n 2^{-n}\pi_i(g_n)x_{i0}u_n$ is $\widetilde{p} = \bigvee_n g_n p g_n^*$ since $\widetilde{x}_{i0}\widetilde{x}_{i0}^*\xi = 0$ iff $g_n x_{i0}x_{i0}^* g_n \xi = 0$ for all n. $\widetilde{p}$ commutes with the g_n and therefore with $\pi_i(\mathcal{L}_I G)'$. Thus, $\widetilde{p} \in \pi_i(\mathcal{L}_I G)' \cap \pi_i(\mathcal{L}_I G)' = \mathbb{C}$ whence $\widetilde{p} = 1$ since $\widetilde{p} \neq 0$.

4th step. It follows that x_{i0} may be chosen to have dense range and by our previous argument, to be injective. A further application of lemma 3.2 then yields a unitary x_{i0}. Notice that the modified $x_{j\square}$ are non–zero. Indeed, by the second braiding relation (3.12) and the unitarity of x_{i0}, the vanishing of $x_{j\square}$ implies that of the original y_{ji}, a contradiction.

Steps 3 and 2 may now be applied to $y_{\square 0}$ yielding an injective operator with dense range. A final application of lemma 3.2 to the diagram (3.13) reflected across the NW–SE diagonal then allows $y_{\square 0}$ to be replaced by its unitary phase. The modified y_{ji} are non–zero since $y_{ji} = \epsilon_j^* x_{j\square} y_{\square 0} x_{i0}^*$ $\diamond$

4. Upper bounds for fusion with the vector representation

PROPOSITION 4.1. *Let $\mathcal{H}_i, \mathcal{H}_\square$ be the positive energy representations at level ℓ with lowest energy subspaces $V_i, V_\square$. Then,*

$$\mathcal{H}_i \boxtimes \mathcal{H}_\square = \bigoplus_j N_{i\square}^j \mathcal{H}_j \tag{4.1}$$

where $\mathcal{H}_j(0) = V_j$ ranges over the summands of $U \otimes V_\square$ which are admissible at level ℓ and $1 \geq N_{i\square}^j \geq 0$. Moreover, $N_{i\square}^j$ vanishes if, and only if the corresponding braiding coefficient λ_j in (3.1) *does.*

PROOF. Let $x_{qp}, y_{qp} : \mathcal{H}_p \to \mathcal{H}_q$ be the intertwiners given by proposition 3.3. Since x_{i0} and $y_{\square 0}$ are unitaries, we have $\mathfrak{X}_i = \mathrm{Hom}_{\mathcal{L}_{I^c}G}(\mathcal{H}_0, \mathcal{H}_i) = x_{i0}\pi_0(\mathcal{L}_{I^c}G)' = x_{i0}\pi_0(\mathcal{L}_I G)''$ and similarly $\mathfrak{Y}_\square = y_{\square 0}\pi_0(\mathcal{L}_I G)''$. Moreover, if $a_1, a_2 \in \pi_0(\mathcal{L}_I G)''$ and $b_1, b_2 \in \pi_0(\mathcal{L}_{I^c}G)''$, we find by (3.12)

$$\langle x_{i0}a_1 \otimes y_{\square 0}b_1, x_{i0}a_2 \otimes y_{\square 0}b_2\rangle = \sum_j \lambda_j \epsilon_j (y_{ji}\pi_i(b_1)x_{i0}a_1\Omega, y_{ji}\pi_i(b_2)x_{i0}a_2\Omega) \tag{4.2}$$

It follows from lemma 3.1 that the map $U : \mathfrak{X}_i \otimes \mathfrak{Y}_\square \to \bigoplus_j \mathcal{H}_j$, $x \otimes y \to \bigoplus_j |\lambda_j|^{\frac{1}{2}} y_{ji}\pi_i(y_{\square 0}^* y)x\Omega$ extends to an isometry $\mathcal{H}_i \boxtimes \mathcal{H}_\square \to \bigoplus_j \mathcal{H}_j$ which is easily seen to commute with the action of

$\mathcal{L}_I G \times \mathcal{L}_{I^c} G$. Notice that the range of U intersects non–trivially all $\mathcal{H}_j$ such that $\lambda_j \neq 0$. Indeed, for any such λ_j, $U(\mathfrak{X}_i \otimes y_{\square 0}) \cap \mathcal{H}_j = y_{ji}\mathfrak{X}_i\Omega$ which is a non–zero subspace of $\mathcal{H}_j$ since $y_{ji} \neq 0$ and $\overline{\mathfrak{X}_i\Omega} = \mathcal{H}_i$. Thus, by the irreducibility of the $\mathcal{H}_j$ under $\mathcal{L}_I G \times \mathcal{L}_{I^c} G$, the image of U is precisely $\bigoplus_{j:\lambda_j \neq 0} \mathcal{H}_j$ $\diamond$

5. Fusion of the vector representation with its symmetric powers

An important corollary of proposition 4.1 and the computations of chapter VIII is the following

PROPOSITION 5.1. *Let $\mathcal{H}_{\theta_1} = \mathcal{H}_{\square}$ and $\mathcal{H}_{k\theta_1}$ be the positive energy representations at level ℓ corresponding to the defining representation V_{θ_1} of SO_{2n} and its k–fold symmetric, traceless power $V_{k\theta_1}$. Then,*

$$\mathcal{H}_{k\theta_1} \boxtimes \mathcal{H}_{\theta_1} = \mathcal{H}_{(k-1)\theta_1} \oplus \mathcal{H}_{k\theta_1+\theta_2} \oplus \mathcal{H}_{(k+1)\theta_1} \tag{5.1}$$

if $1 \leq k \leq \ell - 1$ and

$$\mathcal{H}_{\ell\theta_1} \boxtimes \mathcal{H}_{\theta_1} = \mathcal{H}_{(\ell-1)\theta_1} \tag{5.2}$$

PROOF. By theorem VIII.3.4.1, the constants governing the braiding relations

$$\phi_{k\theta_1\ 0}^{k\theta_1}(w)\phi_{0\ \theta_1}^{\theta_1}(z) = \sum_{\mu \in \{(k-1)\theta_1, k\theta_1+\theta_2, (k+1)\theta_1\}} \beta_{k\theta_1,\mu}\phi_{k\theta_1\ \mu}^{\theta_1}(z)\phi_{\mu\ \theta_1}^{k\theta_1}(w) \tag{5.3}$$

if $k = 1 \dots \ell - 1$ and

$$\phi_{\ell\theta_1\ 0}^{\ell\theta_1}(w)\phi_{0\ \theta_1}^{\theta_1}(z) = \beta_{\ell\theta_1,(\ell-1)\theta_1}\phi_{\ell\theta_1\ (\ell-1)\theta_1}^{\theta_1}(z)\phi_{(\ell-1)\theta_1\ \theta_1}^{\ell\theta_1}(w) \tag{5.4}$$

are non–zero. The fusion rules (5.1)–(5.2) now follow from proposition 4.1 $\diamond$

6. The fusion ring $\mathcal{R}_0$

Let $\mathcal{H}_{\theta_1} = \mathcal{H}_{\square}$ be the vector representation of LG at a fixed level ℓ. We consider below the vector space $\mathcal{R}_0$ generated by the irreducible summands of the iterated fusion products $\mathcal{H}_{\square}^{\boxtimes k}$ and show that it is closed under fusion and forms an associative and commutative ring. The latter property is a direct consequence of the existence of a *braiding* operator B giving a unitary map $\mathcal{H}_i \boxtimes \mathcal{H}_j \to \mathcal{H}_j \boxtimes \mathcal{H}_i$ intertwining LG. A number of other important properties of $\mathcal{R}_0$, most notably the existence of a positive character or quantum dimension function, will be established in section 8.

Let L_0 be the infinitesimal generator of rotations given by the Segal–Sugawara formula (II.1.2.1)

LEMMA 6.1 ([**Wa3**]). *Let $\mathcal{H}_i, \mathcal{H}_j \in \mathcal{P}_\ell$ be such that $\mathcal{H}_j \boxtimes \mathcal{H}_i$ is of positive energy. Then, if $\mathfrak{X}_i, \mathfrak{Y}_j$ are the spaces defined by* (1.1), *the operator $B : \mathfrak{X}_i \otimes \mathfrak{Y}_j \longrightarrow \mathcal{H}_j \boxtimes \mathcal{H}_i$ given by*

$$B(x \otimes y) = e^{-i\pi L_0}\, y^\pi \otimes x^\pi \tag{6.1}$$

where $z^\pi = e^{i\pi L_0} z e^{-i\pi L_0}$ and the $e^{-i\pi L_0}$ on the right hand–side of (6.1) *refers to the positive energy action of* $\mathrm{Rot}\, S^1$ *on $\mathcal{H}_j \boxtimes \mathcal{H}_i$, extends to a unitary $\mathcal{H}_i \boxtimes \mathcal{H}_j \to \mathcal{H}_j \boxtimes \mathcal{H}_i$ intertwining $\mathcal{L}G$.*

PROOF. Notice that B is well–defined since $y^\pi \in \mathfrak{X}_j$ and $x^\pi \in \mathfrak{Y}_i$. Moreover, $L_0\Omega = 0$ and therefore

$$\|y^\pi \otimes x^\pi\|^2 = ((y^\pi)^* y^\pi (x^\pi)^* x^\pi \Omega, \Omega) = (y^* y x^* x e^{-i\pi L_0}\Omega, e^{-i\pi L_0}\Omega) = \|x \otimes y\|^2 \tag{6.2}$$

so that B is norm–preserving and extends to a unitary since it has dense range. As is readily verified, B intertwines the actions of $\mathcal{L}_I G \times \mathcal{L}_{I^c} G$. It follows that $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is of positive energy and, by von Neumann density that B intertwines $\mathcal{L}G$ $\diamond$

DEFINITION. By proposition 4.1, the iterated fusion products of $\mathcal{H}_{\square}$ are finitely reducible positive energy representations. Their irreducible summands therefore generate a vector space $\mathcal{R}_0$ which is closed under fusion and therefore is, by proposition 1.4 and lemma 6.1 an associative and commutative ring.

7. Eigenvalues of the braiding operator

We compute below the eigenvalues of the braiding operator B on $\mathcal{H}_{\square}\boxtimes\mathcal{H}_{\square}$. For convenience, we adopt a graphical notation and label the irreducible SO_{2n}–modules and corresponding positive energy representations by the Young diagram describing their highest weight. Thus, the second and third symmetric, traceless powers of the vector representation $V_{\square}$ of SO_{2n} will be denoted by $V_{\square\square}, V_{\square\square\square}$ and its second and third exterior powers by $V_{\substack{\square\\\square}}, V_{\substack{\square\\\square\\\square}}$. The corresponding positive energy representations are $\mathcal{H}_{\square\square}, \mathcal{H}_{\square\square\square}, \mathcal{H}_{\substack{\square\\\square}}$ and $\mathcal{H}_{\substack{\square\\\square\\\square}}$ respectively. In particular, if the level ℓ is greater or equal to 2, proposition 5.1 yields

$$\mathcal{H}_{\square}\boxtimes\mathcal{H}_{\square}=\mathcal{H}_{\square\square}\oplus\mathcal{H}_{\substack{\square\\\square}}\oplus\mathcal{H}_0 \tag{7.1}$$

The calculation of the eigenvalues of B on (7.1) relies on a version of the braiding relations of proposition 3.3 which is equivariant with respect to rotation by π in the sense that $y_{qp}=x_{qp}^{\pi}=e^{i\pi L_0}x_{qp}e^{-i\pi L_0}$.

Lemma 7.1. *Let $\ell\geq 2$. Then, there exist non–zero bounded operators $x_{qp}\in\mathrm{Hom}_{\mathcal{L}_{I^c}G}(\mathcal{H}_p,\mathcal{H}_q)$ with $x_{\square 0}$ unitary satisfying*

$$x_{\square 0}(x^{\pi}_{\square 0})^*=\sum_j\lambda_j(x^{\pi}_{j\square})^*x_{j\square}\qquad\qquad x_{j\square}x^{\pi}_{\square 0}=\epsilon_j x^{\pi}_{j\square}x_{\square 0} \tag{7.2}$$

where j labels the summands of (7.1) *and $\lambda_j\epsilon_j>0$. Moreover,*

$$\epsilon_j=\sigma_j e^{i\pi(2\Delta_{\square}-\Delta_j)} \tag{7.3}$$

where $\sigma_j=\pm1$ according to whether $\mathcal{H}_j(0)\subset V_{\square}\otimes V_{\square}$ is symmetric or anti–symmetric under $\mathfrak{S}_2$.

Proof. The proof is almost identical to the discussion of section 3 and the proof of proposition 3.3 and we simply indicate the points where they differ. We start from the smeared braiding relations (3.3)–(3.4) where now $i=\square$ labels the vector representation $V_{\square}$ and j the summands of

$$V_{\square}\otimes V_{\square}=V_{\square\square}\oplus V_{\substack{\square\\\square}}\oplus\mathbb{C} \tag{7.4}$$

Let $f\in C^\infty(S^1,V_{\square})$ be supported in $I=(0,\pi)$ and set $g=f^\pi$ where $f^\pi(\theta)=f(\theta-\pi)$. Then, (3.3)–(3.4) read

$$\phi^{\square}_{\square\,0}(f)\phi^{\overline{\square}}_{0\,\square}(\overline{f^\pi})=\sum_j\lambda_j\phi^{\overline{\square}}_{\square\,j}(\overline{f^\pi e_{-\alpha_j}})\phi^{\square}_{j\,\square}(fe_{-\alpha_j}) \tag{7.5}$$

$$\phi^{\square}_{j\,\square}(fe_{-\alpha_j})\phi^{\square}_{\square\,0}(f^\pi)=\epsilon_j\phi^{\square}_{j\,\square}(f^\pi e_{-\alpha_j})\phi^{\square}_{\square\,0}(f) \tag{7.6}$$

where $\alpha_j=2\Delta_{\square}-\Delta_j$. Here, we normalise (7.6) by equating the initial terms of the left–most primary fields so that, by (ii) of lemma VII.3.1, ϵ_j is given by (7.3). By theorem VIII.3.4.1, $\lambda_j\neq 0$ and therefore, by lemma 3.1, $\lambda_j\epsilon_j>0$. Notice that $f^\pi e_{-\alpha_j}=e^{-i\pi\alpha_j}(fe_{-\alpha_j})^\pi$ since f is supported in I. Moreover, by the equivariance of primary fields under the integrally–moded action $R_\theta=e^{i\theta d}$ of $\mathrm{Rot}\,S^1$ (proposition VI.4.1),

$$\phi_{qp}(g^\pi)=R_\pi\phi_{qp}(g)R_{-\pi}=e^{i\pi(\Delta_p-\Delta_q)}e^{i\pi L_0}\phi_{qp}(g)e^{-i\pi L_0}=e^{i\pi(\Delta_p-\Delta_q)}\phi_{qp}(g)^\pi \tag{7.7}$$

and therefore

$$\phi^{\square}_{\square\,0}(f)\phi^{\overline{\square}}_{0\,\square}(\overline{f})^\pi=\sum_j\lambda_j\phi^{\overline{\square}}_{\square\,j}(\overline{fe_{-\alpha_j}})^\pi\phi^{\square}_{j\,\square}(fe_{-\alpha_j}) \tag{7.8}$$

$$\phi^{\square}_{j\,\square}(fe_{-\alpha_j})\phi^{\square}_{\square\,0}(f)^\pi=\epsilon_j\phi^{\square}_{j\,\square}(fe_{-\alpha_j})^\pi\phi^{\square}_{\square\,0}(f) \tag{7.9}$$

Using the adjunction property $\phi^{\overline{\square}}_{q\,p}(\overline{g})=\phi^{\square}_{p\,q}(g)^*$, we find operators x_{qp} satisfying (7.2). The x_{qp} may now be modified in various stages as in the proof of proposition 3.3 with the additional requirement that the x^π_{qp} be altered accordingly. This yields a new set of operators satisfying the same braiding relations but with $x_{\square 0}$ unitary $\diamond$

Proposition 7.2. *Let $\mathcal{H}_\square$ be the vector representation at level $\ell \geq 2$ and $B \in \mathrm{End}_{\mathcal{L}G}(\mathcal{H}_\square \boxtimes \mathcal{H}_\square)$ the braiding operator defined by lemma 6.1. Then, the eigenvalues of B corresponding to the summands of* (7.1) *are distinct and given by*

$$\beta_{\square\square} = q \qquad\qquad \beta_{\substack{\square\\\square}} = -q^{-1} \qquad\qquad \beta_0 = r^{-1} \tag{7.10}$$

where $q = e^{-\frac{i\pi}{\kappa}}$ with $\kappa = \ell + 2(n-1)$ and $r = q^{2n-1}$.

Proof. Let j label the summands of (7.1). By lemma 7.1 and the proof of proposition 4.1, the isomorphism $\mathcal{H}_\square \boxtimes \mathcal{H}_\square \to \bigoplus_j \mathcal{H}_j$ is given by

$$a \otimes b \longrightarrow \bigoplus_j |\lambda_j \epsilon_j|^{\frac{1}{2}} x^\pi_{j\square} \pi_\square((x^\pi_{\square 0})^* b) a\Omega \tag{7.11}$$

for any $a \in \mathfrak{X}_\square = \mathrm{Hom}_{\mathcal{L}_{I^c}G}(\mathcal{H}_0, \mathcal{H}_\square)$ and $b \in \mathfrak{Y}_\square = \mathrm{Hom}_{\mathcal{L}_I G}(\mathcal{H}_0, \mathcal{H}_\square)$. Thus, by (6.1), $B(a \otimes b)$ is mapped to

$$\begin{aligned}
\bigoplus_j |\lambda_j \epsilon_j|^{\frac{1}{2}} e^{-i\pi L_0} x^\pi_{j\square} \pi_\square((x^\pi_{\square 0})^* a^\pi) b^\pi \Omega &= \bigoplus_j |\lambda_j \epsilon_j|^{\frac{1}{2}} e^{-i\pi L_0} x^\pi_{j\square} b^\pi \pi_0((x^\pi_{\square 0})^* a^\pi)\Omega \\
&= \bigoplus_j |\lambda_j \epsilon_j|^{\frac{1}{2}} x_{j\square} b x^*_{\square 0} a\Omega \\
&= \bigoplus_j |\lambda_j \epsilon_j|^{\frac{1}{2}} x_{j\square} x^\pi_{\square 0} (x^\pi_{\square 0})^* b x^*_{\square 0} a\Omega \\
&= \bigoplus_j |\lambda_j \epsilon_j|^{\frac{1}{2}} x_{j\square} x^\pi_{\square 0} x^*_{\square 0} \pi_\square((x^\pi_{\square 0})^* b) a\Omega
\end{aligned} \tag{7.12}$$

which, using (7.2) yields

$$\bigoplus_j \epsilon_j |\lambda_j \epsilon_j|^{\frac{1}{2}} x^\pi_{j\square} \pi_\square((x^\pi_{\square 0})^* b) a\Omega \tag{7.13}$$

Comparing (7.13) with (7.11) we find that the eigenvalues of B are the ϵ_j given by (7.3) where, as customary $\Delta_j = \frac{C_j}{2\kappa}$. Since the Casimir of a representation of highest weight λ is $\langle \lambda, \lambda + 2\rho \rangle$ where $2\rho = 2\sum_{j=1}^n (n-j)\theta_j$ is the sum of the positive roots of Spin_{2n}, we have $C_0 = 0$, $C_\square = 2n-1$, $C_{\square\square} = 4n$ and $C_{\substack{\square\\\square}} = 4(n-1)$ and therefore (7.10) holds $\diamond$

Remark. The eigenvalues of the braiding operator agree with those of the generators of the braid group B_m on m strings when the latter is mapped into the algebra $C_m(q,r)$ defined by Wenzl [**We2**]. Indeed, the latter is the quotient of the group algebra $\mathbb{C}B_m$ with generators g_i, $i = 1 \ldots m-1$ corresponding to the permutation of two succesive strings by a set of relations comprising $(g_i - r^{-1})(g_i + q^{-1})(g_i - q) = 0$. The underlying reason for this is that, for any m, $\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_\square^{\boxtimes m})$ is isomorphic to $C_m(q, q^{2n-1})$, though the proof of this fact requires the knoweldge of the fusion rules with $\mathcal{H}_\square$.

8. The Doplicher–Haag–Roberts theory of superselection sectors

We require some consequences of the theory of superselection sectors, most importantly the existence of a positive homomorphism or *quantum dimension* function on the ring $\mathcal{R}_0$ of section 6, which were originally obtained by Doplicher–Haag–Roberts [**DHR1, DHR2**] and Fredenhagen–Rehren–Schroer [**FRS**]. The precise relation between the predictions of these theories and the bimodule framework for loop groups was first explained in [**Wa4**]. A brief account follows.

Let $\mathcal{H} \in \mathcal{P}_\ell$ be an irreducible positive energy representation of LG and assume that $\mathcal{H}$ has a conjugate, *i.e.* an irreducible $\overline{\mathcal{H}} \in \mathcal{P}_\ell$ such that $\mathcal{H} \boxtimes \overline{\mathcal{H}} \supseteq \mathcal{H}_0$ where $\mathcal{H}_0$ is the vacuum representation. Then, $\overline{\mathcal{H}}$ is unique and $\mathcal{H}_0$ is contained in $\mathcal{H} \boxtimes \overline{\mathcal{H}}$ with multiplicity one. Assume further that all iterated fusion products $\mathcal{H}^{\boxtimes m} \boxtimes \overline{\mathcal{H}}^{\boxtimes n}$ are of positive energy and let $\mathfrak{R}$ be the ring additively generated by their irreducible summands $\mathcal{H}_i$. Then,

- There exists a unique faithful trace on each $\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_{i_1}\boxtimes\cdots\boxtimes\mathcal{H}_{i_m})$ normalised by $\mathrm{tr}(1)=1$ and consistent with the inclusions

$$\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_{i_1}\boxtimes\cdots\boxtimes\mathcal{H}_{i_m})\hookrightarrow\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_{i_1}\boxtimes\cdots\boxtimes\mathcal{H}_{i_m}\boxtimes H_j)\qquad x\to x\boxtimes 1 \tag{8.1}$$
$$\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_{i_1}\boxtimes\cdots\boxtimes\mathcal{H}_{i_m})\hookrightarrow\mathrm{End}_{\mathcal{L}G}(H_j\boxtimes\mathcal{H}_{i_1}\boxtimes\cdots\boxtimes\mathcal{H}_{i_m})\qquad x\to 1\boxtimes x \tag{8.2}$$

- (Jones' relations) Each $\mathcal{H}_i$ possesses a (necessarily unique) conjugate. If $e_k\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_i\boxtimes\overline{\mathcal{H}_i}\boxtimes\mathcal{H}_i)$, $k=1,2$ are the (Jones) projections onto $\mathcal{H}_0\boxtimes\mathcal{H}_i\subseteq(\mathcal{H}_i\boxtimes\overline{\mathcal{H}_i})\boxtimes\mathcal{H}_i$ and $\mathcal{H}_i\boxtimes\mathcal{H}_0\subseteq\mathcal{H}_i\boxtimes(\overline{\mathcal{H}_i}\boxtimes\mathcal{H}_i)$ respectively, then

$$e_1e_2e_1=\tau e_1\qquad\text{where}\qquad\tau=\mathrm{tr}(e_1) \tag{8.3}$$

- (Markov property I) Let $e\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_i\boxtimes\overline{\mathcal{H}_i})$ be the Jones projection onto $\mathcal{H}_0$ and $a\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_j\boxtimes\mathcal{H}_i)$. Then $a_1=a\boxtimes 1$ and $e_2=1\boxtimes e\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_j\boxtimes\mathcal{H}_i\boxtimes\overline{\mathcal{H}_i})$ satisfy

$$\mathrm{tr}(a_1e_2)=\mathrm{tr}(a_1)\,\mathrm{tr}(e_2) \tag{8.4}$$

The above properties are in fact true for any system of bimodules over a type III factor. In our setting however, the following additional properties hold

- (Statistics or braiding operators) There exists a canonical unitary $g\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_i\boxtimes\mathcal{H}_i)$ and the operators $g_1=g\boxtimes 1, g_2=1\boxtimes g\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_i\boxtimes\mathcal{H}_i\boxtimes\mathcal{H}_i)$ satisfy

$$g_1g_2g_1=g_2g_1g_2 \tag{8.5}$$

  Moreover, g coincides with the brading operator B defined by lemma 6.1.
- (Markov property II) Let $g_2=1\boxtimes g\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_j\boxtimes\mathcal{H}_i\boxtimes\mathcal{H}_i)$ be the braiding map corresponding to the last two factors. Then, for any $a\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_j\boxtimes\mathcal{H}_i)$ and $a_1=a\boxtimes 1\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_j\boxtimes\mathcal{H}_i\boxtimes\mathcal{H}_i)$,

$$\mathrm{tr}(a_1g_2^{\pm1})=\mathrm{tr}(a_1)\,\mathrm{tr}(g_2^{\pm1}) \tag{8.6}$$

- (Compatibility) If $g\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_i\boxtimes\mathcal{H}_i)$ and $e\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_i\boxtimes\overline{\mathcal{H}_i})$ are the braiding operator and Jones projection respectively, and $g_1=g\boxtimes 1, e_2=1\boxtimes e\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_i\boxtimes\mathcal{H}_i\boxtimes\overline{\mathcal{H}_i})$, then

$$e_2g_1e_2=\lambda e_2\qquad\text{where}\qquad|\lambda|=|\,\mathrm{tr}(g_2)| \tag{8.7}$$

- (Statistical or quantum dimension) The map

$$d(\mathcal{H}_i)=1/\sqrt{\mathrm{tr}(e)} \tag{8.8}$$

  where $e\in\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_i\boxtimes\overline{\mathcal{H}_i})$ is the Jones projection, extends to a positive homomorphism of the ring $\mathfrak{R}$ into $\mathbb{R}$.

The foregoing discussion and the fusion rules of proposition 5.1 imply

Corollary 8.1. *Let $\mathcal{R}_0$ be the ring generated by the irreducible summands of the iterated fusion products of $\mathcal{H}_\square$ defined in section 6. Then, there exists a positive homomorphism or quantum dimension function $d:\mathcal{R}_0\to\mathbb{R}$.*

Proof. By proposition 5.1, we have $\mathcal{H}_\square\boxtimes\mathcal{H}_\square\supseteq\mathcal{H}_0$ so that $\mathcal{H}_\square$ is self–conjugate and $\mathcal{R}_0$ possesses a positive character $\diamond$

9. The quantum dimension of $\mathcal{H}_\square$ by Wenzl's lemmas

Assuming $\ell\geq 2$, we compute in this section the quantum dimension $d(\mathcal{H}_\square)$ of the vector representation by adapting a calculation of Wenzl [**We3**] to our framework. We proceed as follows. By (8.8) and (8.3), $d(\mathcal{H}_\square)$ may be derived from the Jones relations $e_1e_2e_1=\tau e_1$ satisfied by the projections e_i. To this end, we use a representation of $\mathcal{A}_3=\mathrm{End}_{\mathcal{L}G}(\mathcal{H}_\square\boxtimes\mathcal{H}_\square\boxtimes\mathcal{H}_\square)\ni e_1,e_2$ where the braiding operators g_1,g_2 of (8.5) have a simple form. This gives matrix representatives for the e_i since, by virtue of proposition 7.2, the g_i have distinct eigenvalues and the e_i therefore are spectral projections of the g_i. $d(\mathcal{H}_\square)$ is then obtained by comparing $e_1e_2e_1$ and e_1.

We begin by using our partial knowledge of the fusion rules to describe the structure of $\mathcal{A}_3$. By proposition 5.1,

$$\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square} = \mathcal{H}_{\square\square} \oplus \mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus \mathcal{H}_0 \tag{9.1}$$

and

$$\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square\square} = \mathcal{H}_{\square\square\square} \oplus \mathcal{H}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \oplus \mathcal{H}_{\square} \quad \text{if } \ell \geq 3 \qquad \mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square\square} = \mathcal{H}_{\square} \quad \text{if } \ell = 2 \tag{9.2}$$

Moreover, by proposition 4.1,

$$\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \subseteq \mathcal{H}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \oplus \mathcal{H}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \oplus \mathcal{H}_{\square} \quad \text{if } \ell \geq 3 \qquad \mathcal{H}_{\square} \boxtimes \mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \subseteq \mathcal{H}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \oplus \mathcal{H}_{\square} \quad \text{if } \ell = 2 \tag{9.3}$$

since $V_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ is not admissible at level 2. In fact,

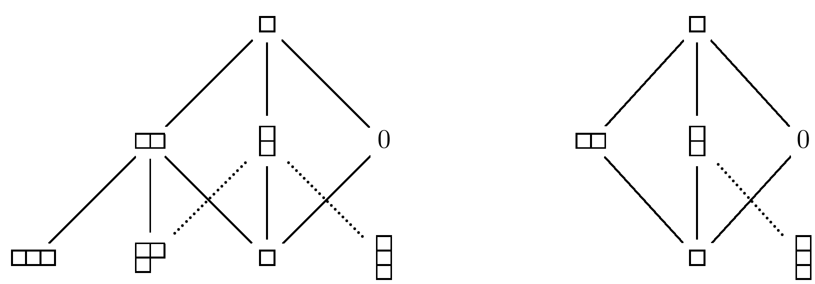
LEMMA 9.1. *If $\ell \geq 2$, then $\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \supseteq \mathcal{H}_{\square}$*

PROOF. From (9.1) we see that $\mathcal{H}_{\square}$ is self–conjugate and therefore that all summands of (9.1) possess conjugates. Conjugating, we find $\overline{\mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}} \subseteq \mathcal{H}_{\square}^{\boxtimes 2}$ and therefore

$$\mathcal{H}_{\square} \boxtimes (\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}) = (\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square}) \boxtimes \mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \supseteq \mathcal{H}_0 \tag{9.4}$$

so that, by uniqueness of conjugates, $\mathcal{H}_{\square} = \overline{\mathcal{H}_{\square}} \subseteq \mathcal{H}_{\square} \boxtimes \mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ $\diamond$

The above fusion rules may be used to describe the inclusion of finite–dimensional algebras $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ where $\mathcal{A}_k = \mathrm{End}_{\mathcal{L}G}(\mathcal{H}_{\square}^{\boxtimes k})$. This is best encoded in a *Bratteli diagram* as follows. Each $\mathcal{A}_k$ is a direct sum of matrix algebras labelled by the inequivalent irreducible summands of $\mathcal{H}_{\square}^{\boxtimes k}$. The corresponding labels are drawn in a row. Lines are drawn from the row corresponding to an algebra to the next, connecting the matrix block of the smaller algebra to each block of the larger which contains it. We therefore get



(9.5)

according to whether $\ell \geq 3$ or $\ell = 2$. The dotted lines indicate that $\mathcal{H}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ or $\mathcal{H}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ may fail to appear in $\mathcal{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \boxtimes \mathcal{H}_{\square}$, in accordance with (9.3). Let now g be the braiding operator on $\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square}$ and e the Jones projection onto $\mathcal{H}_0 \subset \mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square}$. By proposition 7.2, $g = B$ has three distinct eigenvalues $q, -q^{-1}, r^{-1}$ so that e is the spectral projection of g corresponding to r^{-1} and therefore

$$e = \frac{(g-q)(g+q^{-1})}{(r^{-1}-q)(r^{-1}+q^{-1})} \tag{9.6}$$

The same relation binds $g_i, e_i \in \mathcal{A}_3$ where $e_1 = e \boxtimes 1$ and $e_2 = 1 \boxtimes e$ are the Jones projections onto $\mathcal{H}_0 \boxtimes \mathcal{H}_{\square} \subset (\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square}) \boxtimes \mathcal{H}_{\square}$ and $\mathcal{H}_{\square} \boxtimes \mathcal{H}_0 \subset \mathcal{H}_{\square} \boxtimes (\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square})$ respectively and $g_1 = g \boxtimes 1, g_2 = 1 \boxtimes g$ are the braiding operators on $\mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square} \boxtimes \mathcal{H}_{\square}$. Since $\mathcal{H}_0 \boxtimes \mathcal{H}_{\square} \cong \mathcal{H}_{\square} \cong \mathcal{H}_{\square} \boxtimes \mathcal{H}_0$ is irreducible, the e_i are minimal projections in $\mathcal{A}_3$ and $e_i \leq p_{\square}$ where the latter is the minimal central projection of $\mathcal{A}_3$ corresponding to the matrix block labelled by the vector representation. By minimality of e_2, the left action of $\mathcal{A}_3$ on $\mathcal{A}_3 e_2$ is an irreducible representation of $\mathcal{A}_3$ which is faithful on $p_{\square} \mathcal{A}_3 p_{\square}$ and is therefore three–dimensional. We shall use it to find explicit matrix representatives of the e_i. The following gives a convenient basis of $\mathcal{A}_3 e_2$

LEMMA 9.2 (Wenzl,[**We3**]). *If $\ell \geq 2$ then $e_2, e_1e_2, g_1e_2 \in \mathcal{A}_3 = \mathrm{End}_{\mathcal{L}G}(\mathcal{H}_{\square}^{\boxtimes 3})$ are linearly independent and therefore form a basis of the left $\mathcal{A}_3$–module $\mathcal{A}_3e_2$.*

PROOF. Notice first that $1, e_1, g_1$ are linearly independent since g_1 has three dictinct eigenvalues $q, -q^{-1}, r^{-1}$ and e_1 is the spectral projection corresponding to r^{-1}. Let now $\alpha, \beta, \gamma \in \mathbb{C}$ be such that $\alpha e_2 + \beta e_1e_2 + \gamma g_1e_2 = 0$. Multipliplying by $e_2(\alpha + \beta e_1 + \gamma g_1)^*$, we find

$$e_2(\alpha + \beta e_1 + \gamma g_1)^*(\alpha + \beta e_1 + \gamma g_1)e_2 = 0 \tag{9.7}$$

taking traces and using the Markov property (8.4) yields

$$\begin{aligned}\mathrm{tr}(e_2(\alpha + \beta e_1 + \gamma g_1)^*(\alpha + \beta e_1 + \gamma g_1)e_2) &= \mathrm{tr}((\alpha + \beta e_1 + \gamma g_1)^*(\alpha + \beta e_1 + \gamma g_1)e_2)\\ &= \mathrm{tr}((\alpha + \beta e_1 + \gamma g_1)^*(\alpha + \beta e_1 + \gamma g_1))\,\mathrm{tr}(e_2)\\ &= 0\end{aligned} \tag{9.8}$$

and therefore, by faithfulness of the trace $\alpha + \beta e_1 + \gamma g_1 = 0$ whence $\alpha = \beta = \gamma = 0$ $\diamond$

PROPOSITION 9.3 (Wenzl, [**We3**]). *If $\ell \geq 2$, then $e_1e_2e_1 = \tau e_1$ where $\tau = \big(1 + \dfrac{r - r^{-1}}{q - q^{-1}}\big)^{-2}$.*

PROOF. We begin by computing the matrices representing the braiding operators g_1, g_2 on $\mathcal{A}_3e_2$ in the basis e_2, g_1e_2, e_1e_2 of lemma 9.2. The first columns of g_1 and g_2 and the last column of g_1 are obvious since $g_ie_i = r^{-1}e_i$. The second column of g_1 may be computed by expressing g_1^2 in terms of e_1, g_1 via (9.6). We find

$$g_i^2 = 1 + g_i(q - q^{-1}) - zr^{-1}(q - q^{-1})e_i \qquad \text{where} \qquad z = \Big(1 + \frac{r - r^{-1}}{q - q^{-1}}\Big) \tag{9.9}$$

It will be more convenient to work with $c_i = ze_i$. It follows from the above that in the basis c_2, g_1c_2, c_1c_2 of $\mathcal{A}_3e_2$, g_1 and g_2 are given by

$$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & q - q^{-1} & 0 \\ 0 & -r^{-1}(q - q^{-1}) & r^{-1} \end{pmatrix} \qquad \text{and} \qquad g_2 = \begin{pmatrix} r^{-1} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \tag{9.10}$$

The a_{ij} may be determined from the braid relation $g_1g_2g_1 = g_2g_1g_2$. Indeed, comparing the first columns of the two products we find $a_{12} = a_{22} = 0$. Since the braid relations imply that g_1 and g_2 are conjugate, we find by equating their traces and determinants that $a_{33} = q - q^{-1}$ and $a_{23}a_{32} = 1$. The other entries are now easily found by imposing the braid relations and yield

$$g_2 = \begin{pmatrix} r^{-1} & 0 & -(q - q^{-1}) \\ 0 & 0 & 1 \\ 0 & 1 & q - q^{-1} \end{pmatrix} \tag{9.11}$$

From (9.9) and $c_i = ze_i$ we therefore find

$$c_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & r^{-1} & z \end{pmatrix} \qquad \text{and} \qquad c_2 = \begin{pmatrix} z & r & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.12}$$

so that $c_1c_2c_1 = c_1$ and therefore $e_1e_2e_1 = z^{-2}e_1$ as claimed $\diamond$

COROLLARY 9.4. *If $\ell \geq 2$, the quantum dimension of $\mathcal{H}_{\square}$ is equal to $1 + \dfrac{\sin(\frac{(2n-1)\pi}{\kappa})}{\sin(\frac{\pi}{\kappa})}$ where $\kappa = \ell + 2(n - 1)$.*

PROOF. From (8.8), (8.3) and the previous proposition, we find $d(\mathcal{H}_{\square}) = \big(1 + \dfrac{r - r^{-1}}{q - q^{-1}}\big)$ where q, r are given by proposition 7.2 $\diamond$

REMARK. Observe that as $\kappa \to \infty$, $d(\mathcal{H}_{\square}) \to 2n$ which is the dimension of the vector representation of SO_{2n}.

10. The Verlinde algebra

We give below an algebraic characterisation of the *Verlinde rules* which conjecturally describe the fusion of a positive energy representation of $L\,\mathrm{Spin}_{2n}$ with the vector representation $\mathcal{H}_{\square}$. This will be used in the next section to prove that these rules actually hold.

We begin with a heuristic digression. Let $\mathcal{P}_\ell$ be the set of positive energy representations of LG at level ℓ and $\mathcal{A}_\ell$ the alcove parametrising the irreducibles in $\mathcal{P}_\ell$. We define below a set of functions ϕ_μ on $\mathcal{P}_\ell$ indexed by $\mu \in \mathcal{A}_\ell$ which are to be thought of as the characters of the putative $*$-algebra $\mathbb{C}[\mathcal{P}_\ell]$ whose multiplication and involution are defined by fusion and conjugation. If $\lambda \in \mathcal{A}_\ell$ and N_λ is the matrix giving the fusion with $\mathcal{H}_\lambda$, *i.e.*

$$\mathcal{H}_\lambda \boxtimes \mathcal{H}_\mu = \bigoplus_\nu N_{\lambda\mu}^\nu \mathcal{H}_\nu \tag{10.1}$$

then applying a character ϕ of $\mathbb{C}[\mathcal{P}_\ell]$ to both sides of (10.1), we find $\sum_\nu N_{\lambda\mu}^\nu \phi(\mathcal{H}_\nu) = \phi(\mathcal{H}_\lambda)\phi(\mathcal{H}_\mu)$ so that the vector ϕ with entries $\phi(\mathcal{H}.)$ is an eigenvector of N_λ with eigenvalue $\phi(\mathcal{H}_\lambda)$. We now prove this assertion when λ is a minimal weight and N_λ is given by the *Verlinde rules*

Proposition 10.1. *For any integral weight μ in the level ℓ alcove, define a function $\phi_\mu : \mathcal{P}_\ell \to \mathbb{C}$ by $\phi_\mu(\mathcal{H}_\nu) = \chi_\nu(S_\mu)$ where χ_ν is the character of the representation V_ν and $S_\mu = \exp_T(2\pi i \frac{\mu+\rho}{\kappa})$ where $\kappa = \ell + \frac{C_\mathfrak{g}}{2}$. If λ is a dominant minimal weight and N_λ is the matrix defined by*

$$N_{\lambda\,\delta}^\nu = \begin{cases} 1 & \text{if } V_\nu \subseteq V_\lambda \otimes V_\delta \text{ and } \langle \nu, \theta\rangle \leq \ell \\ 0 & \text{otherwise} \end{cases} \tag{10.2}$$

then, for any $\delta \in \mathcal{A}_\ell$, $\sum_\nu N_{\lambda\delta}^\nu \phi_\mu(\mathcal{H}_\nu) = \phi_\mu(\mathcal{H}_\lambda)\phi_\mu(\mathcal{H}_\delta)$ so that the vector with entries $\phi(\mathcal{H}.)$ is an eigenvector of N_λ with eigenvalue $\phi_\mu(\mathcal{H}_\lambda)$.

Proof. Let $\delta \in \mathcal{A}_\ell$, then $\chi_\lambda\chi_\delta = \sum_\nu \chi_\nu$ where ν spans the highest weights of the irreducible summands $V_\nu \subset V_\lambda \otimes V_\delta$. By the tensor product rules with minimal representations (proposition I.2.2.2), these are of the form $\lambda' + \delta$ where $\lambda' \in W\lambda$ is a weight of V_λ and therefore satisfies $|\langle \lambda', \theta\rangle| \leq 1$ so that $\langle \nu, \theta\rangle \leq \ell + 1$. We need to prove that $\chi_\nu(S_\mu) = 0$ whenever $\langle \nu, \theta\rangle = \ell + 1$. By the Weyl character formula, $\chi_\nu = \frac{A_{\nu+\rho}}{A_\rho}$ where 2ρ is the sum of the positive roots of $\mathfrak{g}_\mathbb{C}$ and $A_\alpha(\exp(\beta)) = \sum_w (-1)^w e^{\langle w\alpha, \beta\rangle}$. If $\langle \nu, \theta\rangle = \ell + 1$, then $\nu + \rho$ lies on the affine hyperplane $H_\kappa = \{h | \langle h, \theta\rangle = \kappa\}$ since $\kappa = \ell + \frac{C_\mathfrak{g}}{2} = \ell + 1 + \langle \rho, \theta\rangle$. The affine reflection corresponding to H_κ is $t \to \sigma_\theta(t) + \kappa\theta$ where σ_θ is the orthogonal reflection determined by θ, and therefore, by the integrality of μ, ρ

$$\begin{aligned} A_{\nu+\rho}(\exp(2\pi i \frac{\mu+\rho}{\kappa})) &= A_{\sigma_\theta(\nu+\rho)+\kappa\theta}(\exp(2\pi i \frac{\mu+\rho}{\kappa})) \\ &= \sum_{w \in W} (-1)^w e^{2\pi i \frac{\langle w\sigma_\theta(\nu+\rho), \mu+\rho\rangle}{\kappa}} e^{2\pi i \langle w\theta, \mu+\rho\rangle} \\ &= (-1)^{\sigma_\theta} A_{\nu+\rho}(\mu+\rho) \end{aligned} \tag{10.3}$$

so that $\chi_\nu(S_\mu) = 0$ as claimed $\diamond$

When $G = \mathrm{Spin}_{2n}$ and λ is the highest weight of the vector representation of SO_{2n} with corresponding Young diagram $\square$, we find

Corollary 10.2. *Let $V_\square \cong \mathbb{C}^{2n}$ be the vector representation of SO_{2n} and $N_\square$ the matrix with entries labelled by the irreducible Spin_{2n}–modules which are admissible at level ℓ with $N_{U\square}^W = 1$ if $W \subset U \otimes V_\square$ and 0 otherwise. Then, ϕ_0 is an eigenvector of $N_\square$ with eigenvalue $1 + \dfrac{\sin\frac{(2n-1)\pi}{\kappa}}{\sin\frac{\pi}{\kappa}}$ where $\kappa = \ell + 2(n-1)$.*

PROOF. The dual Coxeter number $\frac{C_{\mathfrak{g}}}{2}$ of SO_{2n} is $2(n-1)$ and we simply need to compute $\chi_{\square}(\exp(2\pi i\frac{\rho}{\kappa}))$. Since the weights of $V_{\square}$ are $\pm\theta_j$, $j=1\dots n$ and $\rho=\sum_{j=1}^n(n-j)\theta_j$, we get

$$\begin{aligned}\chi_{\square}(\exp(2\pi i\frac{\rho}{\kappa})) &= \sum_{j=1}^n\Big(e^{2\pi i\frac{j-n}{\kappa}}+e^{-2\pi i\frac{j-n}{\kappa}}\Big)=2\sum_{j=0}^{n-1}\Re e^{2\pi i\frac{j}{\kappa}}\\ &=2\cos(\frac{(n-1)\pi}{\kappa})\frac{\sin(\frac{n\pi}{\kappa})}{\sin(\frac{\pi}{\kappa})}=1+\frac{\sin(\frac{(2n-1)\pi}{\kappa})}{\sin(\frac{\pi}{\kappa})}\end{aligned} \tag{10.4}$$

◇

We shall need the following well–known formula, see [**Ka2**] for instance

LEMMA 10.3. *If ϕ_0 is defined as in proposition 10.1 by $\phi_0(\mathcal{H}_\lambda)=\chi_\lambda(\exp(2\pi i\frac{\rho}{\kappa}))$, then*

$$\phi_0(\mathcal{H}_\lambda)=\prod_{\alpha>0}\frac{\sin(\frac{\pi\langle\lambda+\rho,\alpha\rangle}{\kappa})}{\sin(\frac{\pi\langle\rho,\alpha\rangle}{\kappa})}>0 \tag{10.5}$$

PROOF. By the Weyl character formula, $\chi_\mu=\dfrac{A_{\mu+\rho}}{\prod_{\alpha>0}(e_{\frac{1}{2}\alpha}-e_{-\frac{1}{2}\alpha})}$ where

$$A_\nu(\exp(\delta))=\sum_{w\in W}e^{\langle w\nu,\delta\rangle}=A_\delta(\exp(\nu)) \tag{10.6}$$

and $e_\beta(\exp(\delta))=e^{\langle\beta,\delta\rangle}$. Setting $\mu=0$, we find Weyl's denominator formula $A_\rho(\exp(2\pi i t))=\prod_{\alpha>0}2i\sin(\pi\langle t,\alpha\rangle)$. Thus,

$$\phi_0(\mathcal{H}_\lambda)=\chi_\lambda(\exp(2\pi i\frac{\rho}{\kappa}))=\frac{A_{\lambda+\rho}}{A_\rho}(\exp(2\pi i\frac{\rho}{\kappa}))=\frac{A_\rho(\exp(2\pi i\frac{\lambda+\rho}{\kappa}))}{A_\rho(\exp(2\pi i\frac{\rho}{\kappa}))}=\prod_{\alpha>0}\frac{\sin(\frac{\pi\langle\lambda+\rho,\alpha\rangle}{\kappa})}{\sin(\frac{\pi\langle\rho,\alpha\rangle}{\kappa})} \tag{10.7}$$

to conclude, notice simply that

$$\langle\rho,\alpha\rangle\leq\langle\rho+\lambda,\alpha\rangle\leq\langle\rho+\lambda,\theta\rangle<1+\langle\rho,\theta\rangle+\ell=\kappa \tag{10.8}$$

and therefore (10.5) holds ◇

11. Main results

Recall the following well–known result [**GM**, chap. 13] whose use is typical in subfactor theory,

THEOREM 11.1 (Perron–Frobenius). *Let N be an irreducible matrix with non–negative entries. Then*

(i) *N has an eigenvector with positive entries. Up to multiplication by a positive constant, the latter is the unique eigenvector of N with non–negative entries. The corresponding (positive) eigenvalue is called the Perron–Frobenius eigenvalue of N.*

(ii) *Let $\widetilde{N}$ be a matrix with non–negative entries bounded above by those of N. Then, any positive eigenvalue of $\widetilde{N}$ is bounded above by the Perron–Frobenius eigenvalue of N. Equality holds only if $\widetilde{N}=N$.*

THEOREM 11.2. *Let $\mathcal{H}_{\square}$ be the vector representation of $L\,\mathrm{Spin}_{2n}$ at level ℓ. Then, for any irreducible, positive energy representation $\mathcal{H}_U$ of $L\,\mathrm{Spin}_{2n}$ whose lowest energy subspace U is a single–valued representation of SO_{2n}, the following holds*

$$\mathcal{H}_U\boxtimes\mathcal{H}_{\square}=\bigoplus_{W\subset U\otimes V_{\square}}N_{U\square}^W\mathcal{H}_W \tag{11.1}$$

where $N_{U\square}^W$ is 1 if W is admissible at level ℓ and 0 otherwise. Any such $\mathcal{H}_U$ appears as a summand in one of the iterated fusion powers $\mathcal{H}_{\square}^{\boxtimes k}$. In particular, it has a conjugate representation and its quantum dimension is given by

$$d(\mathcal{H}_U)=\chi_U(\exp(\frac{2\pi i\rho}{\kappa}))=\prod_{\alpha>0}\frac{\sin(\pi\frac{\langle\lambda+\rho,\alpha\rangle}{\kappa})}{\sin(\pi\frac{\langle\rho,\alpha\rangle}{\kappa})} \tag{11.2}$$

where χ_U and λ are the character of U and its highest weight, the α are the positive roots of Spin_{2n}*,* $2\rho = \sum_{\alpha>0}\alpha$ *and* $\kappa = \ell + 2(n-1)$.

PROOF. If $\ell = 1$, $\mathcal{H}_U$ is either $\mathcal{H}_0$ or $\mathcal{H}_{\square}$ and (11.1) follows from theorem 2.3. The existence of a conjugate is obvious from the fusion rules (2.3) which show that the corresponding Jones projection is 1 so that $d(\mathcal{H}_U) = 1$. This clearly agrees with (11.2) when U is the trivial representation. When $U = V_{\square}$, (10.4) shows that the right–hand side of (11.2) is 1 since $\kappa = 2n-1$ and the theorem is proved.

Let now $\ell \geq 2$. By proposition 4.1, (11.1) holds when the $N_{U\square}^W$ are replaced by some $0 \leq \widetilde{N}_{U\square}^W \leq N_{U\square}^W$, since part of the braiding coefficients involved in the computation of fusion might vanish. Let $N_{\square}$ be the matrix whose entries are the $N_{U\square}^W$, where U and W are single–valued representations of SO_{2n} which are admissible at level ℓ, or equivalently the diagonal block of the matrix defined in corollary 10.2 labelled by these representations. $N_{\square}$ is non–negative and irreducible. Indeed, if U, W are two such representations with highest weights λ, μ so that $\langle\lambda,\theta\rangle, \langle\mu,\theta\rangle \leq \ell$, there exists a sequence of dominant integral weights of SO_{2n}, $\lambda = \nu_1 \cdots \nu_k = \mu$ such that $\langle\nu_j,\theta\rangle \leq \ell$ and each ν_{j+1} is obtained by either adding or removing a box from the Young diagram of μ_j. By the tensor product rules with the vector representation $V_{\square}$ (proposition I.2.2.2), this implies that $W \subseteq U \otimes V_{\square}^{\otimes(k-1)}$ and therefore that $N_{\square}$ is irreducible. By theorem 11.1, $N_{\square}$ has, up to a multiplicative constant, a unique eigenvector with positive entries. On the other hand, by proposition 10.1 and lemma 10.3, the numbers $\phi_0(\mathcal{H}_U) := \phi_0(\mathcal{H}_\lambda)$ where λ is the highest weight of U, are positive and obey

$$\sum_W N_{\square\, U}^W \phi_0(\mathcal{H}_W) = \phi_0(\mathcal{H}_{\square})\phi_0(\mathcal{H}_U) \tag{11.3}$$

It follows that the Perron–Frobenius eigenvector of $N_{\square}$ has entries $\phi_0(\mathcal{H}.)$ and corresponding eigenvalue $\phi_0(\mathcal{H}_{\square})$.

A similar statement about the matrix $\widetilde{N}_{\square}$ may be obtained by using the quantum dimension d of corollary 8.1 on the fusion ring $\mathcal{R}_0$ generated by the irreducible summands of the fusion powers $\mathcal{H}_{\square}^k$. Applying d to (11.1), with $N_{U\square}^W$ replaced by $\widetilde{N}_{U\square}^W$ yields

$$\sum_{W\subset V_{\square}\otimes U} \widetilde{N}_{\square\, U}^W d(\mathcal{H}_W) = d(\mathcal{H}_{\square})d(\mathcal{H}_U) \tag{11.4}$$

whenever $\mathcal{H}_U \in \mathcal{R}_0$. We do not know $\widetilde{N}_{\square}$ to be irreducible, nor is (11.4) a statement about $\widetilde{N}_{\square}$ since some $\mathcal{H}_U$ may not lie in $\mathcal{R}_0$ but if $M_{\square}$ is the matrix whose entries are $\widetilde{N}_{U\square}^W$ if $\mathcal{H}_U, \mathcal{H}_W \in \mathcal{R}_0$ and zero otherwise, we clearly have $M_{\square} \leq \widetilde{N}_{\square} \leq N_{\square}$ entry–wise and therefore, by theorem 11.1 $d(\mathcal{H}_{\square}) \leq \phi_0(\mathcal{H}_{\square})$ with equality only if $M_{\square} = \widetilde{N}_{\square} = N_{\square}$. However, by corollaries 9.4 and 10.2

$$d(\mathcal{H}_{\square}) = 1 + \frac{\sin\frac{(2n-1)\pi}{\kappa}}{\sin\frac{\pi}{\kappa}} = \phi_0(\mathcal{H}_{\square}) \tag{11.5}$$

whence $\widetilde{N}_{\square} = N_{\square}$ and (11.1) holds. The fact that any $\mathcal{H}_U$ appears in some $\mathcal{H}_{\square}^{\boxtimes k}$ now follows from the irreducibility of $\widetilde{N}_{\square} = N_{\square}$. Finally, the claim on the quantum dimension of the $\mathcal{H}_U$ is a direct consequence of the uniqueness of the Perron–Frobenius eigenvector of $N_{\square}$, (11.5) and (10.5) $\diamond$

THEOREM 11.3. *The positive energy representations of $L\,\mathrm{Spin}_{2n}$ at level ℓ whose lowest energy subspace is a single–valued representation of* SO_{2n} *are closed under fusion. They form a commutative and associative ring.*

PROOF. By theorem 11.2, the positive energy representations of $L\,\mathrm{Spin}_{2n}$ corresponding to single–valued representations of SO_{2n} are exactly the irreducible summands of the iterated fusion powers $\mathcal{H}_{\square}^{\boxtimes k}$ of the vector representation and therefore the generators of the commutative and associative ring $\mathcal{R}_0$ of section 6 $\diamond$

The restriction to single–valued representations of SO_{2n} in theorems 11.2 and 11.3 is technical rather than conceptual and these results conjecturally hold for all positive energy representations of $L\,\mathrm{Spin}_{2n}$. At *odd* level, this can be shown by using the discontinuous loops of section I.3.

THEOREM 11.4. *Let $\mathcal{H}_{\square}$ be the vector representation of $L\,\mathrm{Spin}_{2n}$ at odd level ℓ. Then, for any irreducible positive energy representation $\mathcal{H}_U$ with lowest energy subspace U*

$$\mathcal{H}_U \boxtimes \mathcal{H}_{\square} = \bigoplus_{W\subset U\otimes V_{\square}} N_{U\,\square}^{W}\mathcal{H}_W \tag{11.6}$$

where $N_{U\,\square}^{W}$ is 1 if W is admissible at level ℓ and zero otherwise.

PROOF. Notice that the fusion rules (11.6) for $\ell = 1$ were established in theorem 2.3. Let now U be a single–valued SO_{2n}–module so that (11.6) holds by theorem 11.2. If ζ is a discontinuous loop, proposition 2.2 yields $\zeta(\mathcal{H}_U \boxtimes \mathcal{H}_{\square}) \cong (\zeta\mathcal{H}_0 \boxtimes \mathcal{H}_U) \boxtimes \mathcal{H}_{\square} \cong \mathcal{H}_{\zeta(U)} \boxtimes \mathcal{H}_{\square}$ where the notation refers to the action of $Z(\mathrm{Spin}_{2n})$ on the irreducible Spin_{2n}–modules admissible at level ℓ given by proposition I.3.1.5. Conjugating both sides of (11.6), we find

$$\mathcal{H}_{\zeta(U)} \boxtimes \mathcal{H}_{\square} = \bigoplus_{\substack{W\subset U\otimes V_{\square}\\ W\ \ell\text{-admissible}}} \mathcal{H}_{\zeta(W)} \tag{11.7}$$

We claim that $W \subseteq U \otimes V_{\square}$ if, and only if $\zeta(W) \subseteq \zeta(U) \otimes V_{\square}$. Assuming this for a moment, (11.7) may be rewritten as

$$\mathcal{H}_{\zeta(U)} \boxtimes \mathcal{H}_{\square} = \bigoplus_{\substack{X\subset \zeta(U)\otimes V_{\square}\\ X\ \ell\text{-admissible}}} \mathcal{H}_{X} \tag{11.8}$$

so that (11.6) holds for $\zeta\mathcal{H}_U$. Since ℓ is odd, corollary I.3.2.6 implies that any irreducible representation of $L\,\mathrm{Spin}_{2n}$ at level ℓ is of the form $\zeta\mathcal{H}_U$ for a suitable ζ and a single–valued SO_{2n}–module U and theorem 11.4 is proved. Returning to our claim, let $\lambda_W, \lambda_{\square}, \lambda_U$ be the highest weights of $W, V_{\square}$ and U respectively. By the tensor product rules with minimal representations (proposition I.2.2.2), $W \subseteq U \otimes V_{\square}$ iff $\lambda_W = \lambda_U + w\lambda_{\square}$ for some $w \in W$. By proposition I.3.1.5, $\zeta(\lambda) = w_i\lambda + \ell\lambda_i^{\vee}$, and we find that this is the case iff $\zeta(\lambda_W) = \zeta(\lambda_U) + w'\lambda_{\square}$ for some $w' \in W$ and therefore iff $\zeta(W) \subseteq \zeta(U) \otimes V_{\square}$ $\diamond$

THEOREM 11.5. *At odd level, the positive energy representations of $L\,\mathrm{Spin}_{2n}$ are closed under fusion and have conjugates. They form a commutative and associative ring.*

PROOF. Let U_i, $i = 1, 2$ be single–valued representations of SO_{2n}. By theorem 11.3, $\mathcal{H}_{U_1} \boxtimes \mathcal{H}_{U_2}$ is of positive energy. Applying discontinuous loops ζ_1, ζ_2, we find that

$$\zeta_1\mathcal{H}_{U_1} \boxtimes \zeta_2\mathcal{H}_{U_2} \cong \zeta_1\mathcal{H}_0 \boxtimes (\mathcal{H}_{U_1} \boxtimes \zeta_2\mathcal{H}_{U_2}) \boxtimes \zeta_2\mathcal{H}_0 \cong \zeta_1\zeta_2(\mathcal{H}_{U_1} \boxtimes \zeta_2\mathcal{H}_{U_2}) \tag{11.9}$$

is of positive energy. By corollary I.3.2.6, any irreducible representation at odd level is of the form $\zeta\mathcal{H}_U$ for some ζ and U single–valued and our first claim is proved. The existence of conjugates follows similarly for $\mathcal{H}_U \boxtimes \overline{\mathcal{H}_U} \supseteq \mathcal{H}_0$ for U single–valued implies that $\zeta\mathcal{H}_U \boxtimes \zeta^{-1}\overline{\mathcal{H}_U} \supset \mathcal{H}_0$ and therefore that $\zeta\mathcal{H}_U$ has a conjugate. The associativity and commutativity are immediate consequences of the associativity of fusion and the existence of braiding $\diamond$

REMARK. It is readily verified that both sides of (11.2) are invariant under the action of $Z(\mathrm{Spin}_{2n})$ and therefore that the quantum dimension of all irreducible positive representations of $L\,\mathrm{Spin}_{2n}$ at odd level is given by the formula (11.2).

Concluding remarks

1. Computation of the fusion ring of $L\,\mathrm{Spin}_{2n}$

The structure of the level ℓ fusion ring $R_\ell(LG)$ has been known conjecturally for some time. It should be isomorphic to the quotient of the representation ring $R(G)$ by a certain 'holomorphic induction' map i_ℓ taking V to $\mathcal{H}_V$ whenever V is admissible at level ℓ. One way to prove this is to show that

$$i_\ell(V) \boxtimes i_\ell(U) = i_\ell(V \otimes U) \tag{1.1}$$

for all fundamental representations V since these generate $R(G)$ as a ring. For $G = \mathrm{Spin}_{2n}$, they are the exterior powers $\Lambda^k V_\square$ of the vector representation $V_\square$ with $k = 1 \ldots n-2$ and the spin modules $V_{s_\pm}$. The fusion rule (1.1) for $V = V_\square$ was proved in chapter IX. In fact, for *odd* ℓ

$$\mathcal{H}_\square \boxtimes \mathcal{H}_U = \bigoplus_W N_{\square U}^W \mathcal{H}_W \tag{1.2}$$

where W spans the summands of $V_\square \otimes U$ and $N_{\square U}^W$ is 1 or 0 according to whether W is admissible at level ℓ or not. For even ℓ, (1.2) was shown only for single–valued SO_{2n}–modules U. This restriction will be lifted in §1.1.

A. Wassermann has outlined a method for computing the remaining fusion rules, thereby completing the calculation of $R_\ell(L\,\mathrm{Spin}_{2n})$. We reproduce it below.

1.1. Fusing with the spin representations $\mathcal{H}_{s_\pm}$.

Since the spin modules $V_{s_\pm}$ are minimal representations of Spin_{2n}, the corresponding Verlinde rules (1.1) are of the same form as (1.2), namely

$$\mathcal{H}_{s_\pm} \boxtimes \mathcal{H}_U = \bigoplus_{W \subset V_{s_\pm} \otimes U} N_{s_\pm U}^W \mathcal{H}_W \tag{1.1.1}$$

where $N_{s_\pm U}^W$ is 1 if W admissible at level ℓ and 0 otherwise.

The proof of (1.1.1) parallels that of (1.2) given in chapter IX. Briefly, the braiding relations of the spin primary fields with a general primary field are converted, by smearing on test functions with disjoint supports and taking phases, into commutation relations of bounded intertwiners for the local loop groups. Most of the required analysis has been carried out in chapter VI where all the primary fields needed for fusing a general representation with $\mathcal{H}_{s_\pm}$ were shown to extend to operator–valued distributions. A technical point needs to be addressed however. At present, the procedure of taking phases to produce bounded intertwiners applies only if one of the two primary fields extends to a bounded operator–valued distribution. This is the case for the vector primary field, since it is essentially a Fermi field but does not hold for the spin fields. Once this problem is bypassed, the bounded intertwiners give an $L\,\mathrm{Spin}_{2n}$–equivariant isometry of the left hand–side of (1.1.1) into the right hand–side and therefore the upper bound

$$0 \leq \widetilde{N}_{s_\pm U}^W \leq N_{s_\pm U}^W \tag{1.1.2}$$

where the $\widetilde{N}_{s_\pm U}^W$ are the structure constants of fusion. This shows that fusion with the spin representations is always of positive energy and that these representations have (necessarily unique) conjugates.

Indeed, by (1.1.2), $\mathcal{H}_{s_\pm}\boxtimes\mathcal{H}_{s_\pm}$ contains only representations whose lowest energy subspaces are single–valued SO_{2n}–modules. As shown in chapter IX, these representations have conjugate bimodules so that for some $\mathcal{H}_W$,

$$\mathcal{H}_{s_\pm}\boxtimes(\mathcal{H}_{s_\pm}\boxtimes\mathcal{H}_W)=(\mathcal{H}_{s_\pm}\boxtimes\mathcal{H}_{s_\pm})\boxtimes\mathcal{H}_W\supset\mathcal{H}_0 \tag{1.1.3}$$

whence $\mathcal{H}_{s_\pm}$ have conjugates.

We now use the method of chapter IX to prove that (1.2) holds for any representation U of Spin_{2n} admissible at level ℓ. Using the Doplicher–Haag–Robert theory, the quantum dimension d may be extended from the ring $\mathcal{R}_0$ generated by the iterated fusion powers of $\mathcal{H}_\square$ to the ring $\widetilde{\mathcal{R}_0}$ generated by the fusion powers of $\mathcal{H}_\square$ and $\mathcal{H}_{s_\pm}$. Consider the matrix $N_\square$, whose entries $N_{\square U}^W$ are now indexed by two–valued representations of SO_{2n}. $N_\square$ is non–negative, irreducible and, by simple inspection, has a Perron–Frobenius eigenvector whose entries are the 'dimensions'

$$\phi_0(\mathcal{H}_U)=\chi_U(\exp(\frac{2\pi i\rho}{\kappa})) \tag{1.1.4}$$

given in [**Ka1**, §13.8] and studied in chapter IX. The corresponding eigenvalue is $\phi_0(\mathcal{H}_\square)$. The fusion rules with $\mathcal{H}_\square$ read

$$\mathcal{H}_\square\boxtimes\mathcal{H}_U=\bigoplus_{W\subset V_\square\otimes U}\widetilde{N}_{\square U}^W\mathcal{H}_W \tag{1.1.5}$$

where, as shown in chapter IX, $\widetilde{N}_{\square U}^W\leq N_{\square U}^W$ with equality if U is a single–valued SO_{2n}–module. Let the matrix $M_\square$ have entries $M_{\square U}^W=\widetilde{N}_{\square U}^W$ if $U,W\in\widetilde{\mathcal{R}_0}$ and 0 otherwise so that $M_\square\underline{d}=d(\mathcal{H}_0)\underline{d}$ where d is the (non–zero) vector with components $d(\mathcal{H}_U)$. Since $M_\square\leq\widetilde{N}_\square\leq N_\square$ and $d(\mathcal{H}_0)=\phi_0(\mathcal{H}_\square)$, we have $M_\square=N_\square$. Therefore (1.2) holds and all positive energy representations of level ℓ lie in $\widetilde{\mathcal{R}_0}$. Moreover, by uniqueness of Perron–Frobenius eigenvectors, the quantum dimension of those corresponding to two–valued representations of SO_{2n} is given by $d(\mathcal{H}_U)=\gamma\phi_0(\mathcal{H}_U)$ for some $\gamma>0$ while, by the results of chapter IX, $d(\mathcal{H}_U)=\phi_0(\mathcal{H}_U)$ whenever U is a single–valued SO_{2n}–module.

Let now U be a single–valued representation of SO_{2n}, then

$$\mathcal{H}_{s_\pm}\boxtimes\mathcal{H}_U=\bigoplus_{W\subset V_{s_\pm}\otimes U}\widetilde{N}_{s_\pm U}^W\mathcal{H}_W \tag{1.1.6}$$

where all W are two–valued on SO_{2n}. Since $d(\mathcal{H}_{s_\pm})=\gamma\phi_0(\mathcal{H}_{s_\pm})$, $d(\mathcal{H}_U)=\phi_0(\mathcal{H}_U)$ and $d(\mathcal{H}_W)=\gamma\phi_0(\mathcal{H}_W)$, applying d to (1.1.6) we get

$$\sum_{W\subset V_{s_\pm}\otimes U}\widetilde{N}_{s_\pm U}^W\phi_0(\mathcal{H}_W)=\phi_0(\mathcal{H}_{s_\pm})\phi_0(\mathcal{H}_U) \tag{1.1.7}$$

On the other hand, the dimensions $\phi_0(\mathcal{H}.)$ satisfy (1.1.7) when the $\widetilde{N}_{s_\pm U}^W$ are replaced by $N_{s_\pm U}^W$ and therefore the two numbers must be equal by (1.1.2). It follows that (1.1.1) holds whenever U is a single–valued representation of SO_{2n}. To conclude, assume that the conjugate bimodules of $\mathcal{H}_s$ is $\overline{\mathcal{H}_s}=\mathcal{H}_{\overline{s}}$ where s is $s_\pm$ and $\overline{s}$ is the corresponding dual highest weight, so that $\overline{s}$ is $s_\pm$ if n is even and $s_\mp$ if n is odd. Then, if U is a single–valued SO_{2n}–module, we get by Frobenius reciprocity

$$N_{sU}^W=\widetilde{N}_{sU}^W=\dim\mathrm{Hom}_{L\,\mathrm{Spin}_{2n}}(\mathcal{H}_s\boxtimes\mathcal{H}_U,\mathcal{H}_W)=\dim\mathrm{Hom}_{L\,\mathrm{Spin}_{2n}}(\mathcal{H}_U,\mathcal{H}_{\overline{s}}\boxtimes\mathcal{H}_W)=\widetilde{N}_{\overline{s}W}^U \tag{1.1.8}$$

and therefore, using the symmetry of the Verlinde numbers, $\widetilde{N}_{\overline{s}W}^U=N_{sU}^W=N_{\overline{s}W}^U$ which concludes the proof of (1.1.1). To summarise, the above method relies on

(i) Extending the procedure of taking phases to produce the braiding of bounded intertwiners from that of primary fields to the case where both primary fields are unbounded operator–valued distributions.
(ii) Proving that $\overline{\mathcal{H}_s}=\mathcal{H}_{\overline{s}}$.

1.2. Fusing with the exterior powers $\mathcal{H}_{\Lambda^k V}$.

The fusion rules with the exterior powers $\mathcal{H}_{\Lambda^k V}$, $k = 2 \dots n-2$ are more delicate to handle since they involve multiplicities other than 0 or 1. This difficulty can be circumvented by realising the multiplicity spaces as representations of the Birman–Wenzl algebra BW_k on k strings. The latter is relevant because a generalisation of Brauer–Weyl duality [**Br**] holds, namely the commutant of $L\,\mathrm{Spin}_{2n}$ on the k–fold fusion $\mathcal{H}_{\square}^{\boxtimes k}$ is generated by the action of the braid group B_k and this action factors through BW_k. Moreover, $\mathcal{H}_{\Lambda^k V}$ is the isotypical summand of $\mathcal{H}_{\square}^{\boxtimes k}$ corresponding to the quantum sign representation ε_k of B_k. Thus, $\mathcal{H}_{\Lambda^k V} \boxtimes \mathcal{H}_U$ may be computed as the subspace of $\mathcal{H}_{\square}^{\boxtimes k} \boxtimes \mathcal{H}_U$ transforming like ε_k under the natural action of B_k. This may be achieved by realising the action more explicitly, by analogy with the $L\,\mathrm{SU}_n$ case treated in [**Wa3**] on chains of k vector primary fields of the form

$$\phi_{V_k V_{k-1}}(f_k) \cdots \phi_{V_1 U}(f_1) \tag{1.2.1}$$

smeared on functions supported in k consecutive subintervals of $I = (0, 2\pi)$, where the action of B_k is given by braiding. As for the previously computed fusion rules, this analysis gives an upper bound for $\mathcal{H}_{\Lambda^k V} \boxtimes \mathcal{H}_U$ in terms of the Verlinde rules. A Perron–Frobenius argument based on the computation of the quantum dimension of $\mathcal{H}_{\Lambda^k V}$ given in chapter IX then shows that the bound is attained.

2. Loop groups of other Lie groups

The loop groups of Lie groups other than SU_n and Spin_{2n} need to be investigated. Our results extend almost *verbatim* to Spin_{2n+1}. Moreover, the analysis of the continuity properties of primary fields done in chapters V and VI covers the level 1 and a number of higher level fields for all simply–laced groups. The primary fields for the remaining Sp_n, F_4, G_2 may probably be studied via the generalised vertex operator construction [**GNOS**].

3. Loop groups, quantum invariant theory and subfactors

Any positive energy representation $(\pi, \mathcal{H})$ of LG gives rise to two subfactors. The first is obtained via the quantum invariant theory inclusion

$$\Big(\bigcup_m \mathbb{C} \otimes \mathrm{End}_{LG}(\mathcal{H}^{\boxtimes m})\Big)'' \subset \Big(\bigcup_m \mathrm{End}_{LG}(\mathcal{H}^{\boxtimes m})\Big)'' \tag{3.1}$$

and is of type II_1 with the trace given by the Doplicher–Haag–Roberts theory. The second is the Jones–Wassermann inclusion

$$\pi(L_I G)'' \subset \pi(L_{I^c} G)' \tag{3.2}$$

where $L_I G$ is the group of loops supported in the proper interval $I \subset S^1$. As shown by Jones and Wassermann [**Wa1**] and explained in chapter IV, (3.2) is an irreducible inclusion of hyperfinite factors of type III_1 whenever π is irreducible. Using results of Popa [**Po1, Po2**], Wassermann proved that for $G = \mathrm{SU}_n$, (3.2) is isomorphic to the tensor product of the hyperfinite factor of type III_1 with a Jones-Wenzl subfactor [**Wa1**]. The latter is defined using the Hecke algebras $H_m(q)$ with $q = e^{\pi i/(N+\ell)}$ and the representation of SU_n corresponding to the lowest energy subspace of $\mathcal{H}$ [**We1**]. Interestingly, it may also be defined as the quantum invariant theory inclusion corresponding to $\mathcal{H}$. Since the positive energy representations of LG and the subfactors defined in [**We2**] are characterised by the same data for G of type B, C or D, a natural conjecture is that a similar result holds in these cases. For $G = \mathrm{Spin}_{2n}$, this should follow from a computation of the higher relative commutants by showing that $\mathrm{End}_{L\,\mathrm{Spin}_{2n}}(\mathcal{H}_{\square}^{\boxtimes n})$ is isomorphic to the Birman-Wenzl algebra on n strings. In fact, only the II_1 subfactors corresponding to the single–valued representations of SO_{2n} are constructed in [**We2**]. The quantum invariant theory inclusion on the other hand gives subfactors for *any* representation of Spin_{2n}. We intend to study the corresponding spin subfactors, the structure of which should be linked to the delicate combinatorics of the tensor product rules for the two–valued representations of Spin_{2n}.

4. Modular categories and invariants of 3-manifolds

An important class of invariants of 3–manifolds is obtained via surgery on framed links in the 3-sphere. These use *modular tensor categories* which are traditionally produced as the semi–simple quotients of the representation theory of quantum groups at roots of unity [**Tu**]. The analysis involved relies on the combinatorics of the tensor products of the representations of the underlying Lie algebras [**TuWe**]. So far, two main difficulties have prevented the construction of the invariants corresponding to Spin_{2n}. On the one hand, the intricacies of the combinatorics of two–valued representations of SO_{2n} have led researchers to consider the (truncated) representation ring of SO_{2n} only. On the other, the latter gives rise to a braided tensor category which is not modular, in that the corresponding S matrix is not invertible. As a result, only the invariants corresponding to $\mathrm{PSO}_{2n} = \mathrm{SO}_{2n}/\mathbb{Z}_2$ have been defined. The category $\mathcal{P}_\ell$ of positive energy representations of $L\,\mathrm{Spin}_{2n}$ at level ℓ, endowed with Connes fusion should provide a natural solution to this problem. Indeed, the Doplicher–Haag–Roberts theory implies that $\mathcal{P}_\ell$ is a braided tensor category and there remains to verify that $\mathcal{P}_\ell$ is modular.

5. Knizhnik–Zamolodchikov equations

In chapter VIII, we solved the KZ differential equations corresponding to a number of important fusion rules for $L\,\mathrm{Spin}_{2n}$ by using the contour integrals of Dotsenko and Fateev [**DF**]. These have been greatly generalised by Schechtman–Varchenko [**SV**] and Feigin–Frenkel [**EdF**] who used them systematically to give all solutions of the KZ equations. It is interesting to notice however that their generalised hypergeometric solutions, which first appeared in the work of Aomoto and Gelfand [**Ao, GKZ**], use Euler–like contour integrals with a large number of integration variables (typically growing linearly in n) which make them intractable for computational purposes. Our solutions on the other hand use 2 integration variables for any n. There might therefore exist a universal cohomological simplification mechanism applicable to the contour integrals of Schechtman–Varchenko and Feigin–Frenkel which could well lead to a proof of the symmetry properties of the braiding coefficents conjectured by Witten in the context of restricted solid–on–solid statistical mechanical models.

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